Discrete Structures: Homework 3 Hints

You are not required to read or to do anything with these hints. They only address one problem (HW3, P4).

Problem 4 (Rosen2019, #33, p.465) – *After Ch.6.*
How many bit strings of length n, where $n \geq 4$, contain exactly two occurrences of `01`.
In order to develop our counting techniques, we can consider a simpler problem: Find those bit strings of length n (where $n \geq 2$) containing exactly one occurrence of `01`.
Such numbers can be represented like this:

```
.......01.........
```

There `01` denotes the only occurrence of "01", but the sequences of dots represent (possibly empty) non-increasing sequences of k and $n-k-2$ bits. Here $k = 0, 1, 2, \ldots, n-2$. (A sequence of bits is non-increasing, if it is constantly equal to "1" or to "0", or switches from value "1" to "0", but never from "0" to "1").
Statement. There are altogether $k+1$ non-increasing sequences of length k. You can enumerate them (they may contain no more than one location, where the value switches from "1" to "0").
Consequently, as you move the only pattern `01` into various positions k, where ($k = 0, 1, 2, \ldots, n-2$ denotes the *offset* – how many symbols are to the left of this pattern. The total number of such strings is equal to this sum:

$$S_n = \sum_{k=0}^{n-2} (k+1) \cdot (n-k-1). \tag{1}$$

Indeed, there are two non-increasing sequences of length k and $n-k-2$. They can be independently selected in $k+1$ and $(n-k-2)+1$ ways respectively. For example, if $n=5$, we get $S = 1 \cdot 4 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1$.

```
01000
01100
01110
01111
00100
00110
00111
10100
10110
10111
00010
00011
10010
10011
11010
11011
00001
10001
11001
11101
```

Once we know, how to express S_n (the number of bit strings of length n that contain the pattern 01 only once), we can calculate its values:

n	S_n
2	$1 = 1 \cdot 1$
3	$4 = 1 \cdot 2 + 2 \cdot 1$
4	$10 = 1 \cdot 3 + 2 \cdot 2 + 3 \cdot 1$
5	20
6	35
7	56
8	84
9	120
10	165

Statement: If S_n expresses the number of bit strings of length n with exactly one pattern `01`, then the following statement is true:

$$S_{n+1} = S_{n-1} + n^2.$$

We can prove this identity by plugging into **??**
Summation of squares has this formula (can be proven by induction):

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Squares of even numbers have a similar formula (four times bigger):

$$2^2 + 4^2 + \ldots + (2n)^2 = \frac{2n(2n+2)(2n+1)}{6}$$

Therefore, for odd n we can add together expressions like these:

$$S_9 = S_7 + 8^2 = (S_5 + 6^2) + 8^2 = \ldots =$$
$$= 2^2 + 4^2 + 6^2 + 8^2$$

.
For odd n we have a sum of all squares:

$$S_{2n+1} = \frac{2n(2n+2)(2n+1)}{6}.$$

But a similar formula applies to even n as well:

$$S_{2n} = \frac{(2n-1)(2n+1)2n}{6}.$$

So the formula that is valid for any n:

$$S_n = \frac{(n-1)n(n+1)}{6} = \binom{n+1}{3}.$$