Discrete Structures Lab01: Hints

Lemma 1: For all propositions a, $\neg\neg a \to a$.
Proof:

- Assume that $\neg\neg a$ is true.
- We sort two cases by "classic" axiom (Excluded Middle): either a or $\neg a$ must be true.
- If a is true, we are happy.
- Otherwise $\neg a$ is true (along with $\neg\neg a$ obtained before). This is a contradiction.
- Therefore a must be true in all cases (it is either trivial, or a contradiction).

Lemma 2: For all propositions a and b, $\neg(a \to b) \to a$.
Proof:

- Assume that $\neg(a \to b)$ is true.
- We have to prove that a is true. By Lemma 1, we will prove instead that $\neg nega$ is true, then it will also imply a.
- We will assume that $\neg a$ is true, and attempt to get a contradiction (this means that $\neg\neg a$ must be true).
- Let's prove now that $a \to b$ is true - this would be an immediate contradiction with $\neg(a \to b)$.
- To prove $a \to b$, assume that a is true and let's prove b. But earlier we assumed that $\neg a$.
- a and $\neg a$ cannot be simultaneously true. This is a contradiction.

Peirce Lemma: For all propositions a and b, $((a \to b) \to a) \to a$.

Hint. Just use Lemma 1 and 2 for this. And also the "classic" axiom: Sort 2 cases when $(a \to b)$ or $\neg(a \to b)$ are true.

Lemma 4: For all propositions a and b, $(\neg b \to \neg a) \to (a \to b)$.
This is the opposite direction from a well-known contrapositive ($\neg b \to a$ and $a \to b$ mean the same thing.)

Hint. Use "classic" axiom (Excluded middle) on b.

Lemma 5: For all propositions a, b, c, d, e,

$$((((a \to b) \to (\neg c \to \neg d)) \to c) \to e) \to \neg a \to (d- > e).$$

Hint. Indeed, assume that $(((a \to b) \to \neg c \to \neg d) \to c) \to e$; also assume $\neg a$ and d. Then you can prove $(((a \to b) \to \neg c \to \neg d) \to c)$ which is similar to what you need.

After all this, you can do Sample20 from the Coq lab.