Final Exam, 2020-04-23

Question 1.

By U we denote the set of all positive integers between 1 and 120. This is the *universe* in which we define several subsets:

$$\begin{cases} A = \{x \in U \mid 2 \mid x\}, \\ B = \{x \in U \mid 3 \mid x\}, \\ C = \{x \in U \mid 5 \mid x\}, \\ X = \{x \in U \mid 2 \mid x \lor 3 \mid x\}, \\ Y = \{x \in U \mid (3 \mid x \land 5 \mid x) \lor \neg(2 \mid x)\}. \end{cases}$$

(A) Express X using the sets A, B, C (using set union $V \cup W$, set intersection $V \cap W$, set complement $\overline{V}$ operations).
(B) Express Y using the sets A, B, C in a similar way.
(C) Find $|X|$ - the size of the set X.
(D) Find $|Y|$ - the size of the set Y.

Question 2.

Let A and B be sets with sizes $|A| = 8$ and $|B| = 5$ and $|A \cap B| = 3$.
Calculate the largest and the smallest possible values for each of the following set sizes:
(A) $|A \cup B|$.
(B) $|A \times (B \times B)|$.
(C) $|\mathcal{P}(\mathcal{P}(A \cap B))|$ - the powerset of a powerset of $A \cap B$.
(D) $|A \oplus B|$ - the symmetric difference of the sets A and B.

Question 3.

Consider the following recurrent sequence:

$$\begin{cases} a_0 = 3 \\ a_1 = 4 \\ a_{n+2} = 5a_{n+1} - 6a_n, \text{ if } n \geq 0 \end{cases}$$

Assume that b_n is another sequence satisfying the recurrence rule

$$b_{n+2} = 5b_{n+1} - 6b_n, \text{ if } n \geq 0$$

(The first two members b_0, b_1 are not known.)
(A) Write the first 6 members of this sequence $(a_0, \ldots, a_5)$.
(B) Write the characteristic equation for this sequence.
(C) Write the general expression for an arbirary sequence b_n satisfying the recurrent expression as a sum of two geometric progressions (you can leave unknown coefficients in your answer; just explain which ones they are).
(D) Write the formula to compute a_n (that would satisfy the initial conditions $a_0 = 3$ and $a_1 = 4$).

Question 4.

Consider this code snippet in Python:

```
n = 1000
sum = 0
for i in range(1, n*n+1):
    for j in range(1,i+1):
        sum += i % j
```

And a similar one in R:

```
n <- 1000
sum <- 0
for (i in 1:(n*n)) {
    for (j in 1:i) {
        sum <- sum + i %% j
    }
}
```

(A) Explain in human language what this algorithm does.
(B) Denote by $f(n)$ the number of times the variable ‘sum‘ is incremented. Write the Big-O-Notation for $f(n)$. Find a function $g(n)$ such that $f(n)$ is in $O(g(n))$. (If there are multiple functions, pick the one with the slowest growth.)
(C) Express the function $f(n)$ precisely - how many times ‘sum‘ is incremented in terms of variable n.

Question 5.

Let A be the set of all positive divisors of the number 120 (including 1 and 120 itself).
(A) What is the multiplication of all numbers in the set A?
(B) Express this number as the product of prime powers.

Question 6.

Define the following binary relationship on the set of integer numbers $\mathbb{Z}$: We say that aRb (numbers $a, b \in \mathbb{Z}$ are in the relation R) iff

$$\begin{cases} a - b \equiv 0 \pmod{11} \\ a - b \equiv 0 \pmod{12} \\ a - b \equiv 0 \pmod{13} \end{cases}$$

Item	Statement	True or False?
(A)	R is reflexive	
(B)	R is symmetric	
(C)	R is antisymmetric	
(D)	R is transitive	
(E)	aRb iff $a = b$	

For all items where you answered ‘FALSE‘, specify a counterexample (values for some numbers that would make the condition true, but the conclusion false). If the statement was true, write "none".
(A) counterexample: ...
(B) counterexample: ...
(C) counterexample: ...
(D) counterexample: ...
(E) counterexample: ...

Question 7.
Four people A, B, C, D each has his own hat. After the meeting they leave their building in a hurry, everyone grabs some hat at random so that all 4! permutations of the hats have equal probabilities.
Let the random variable X denote the number of hats that were picked up correctly. (For example, if the hat assignment is this: $(A \rightarrow A, B \rightarrow B, C \rightarrow D, D \rightarrow C)$, then $X = 2$, because two people got their own hats.)
(A) Find $E(X)$ - the expected value of X.
(B) Find $V(X)$ - the variance of X.

Question 8.
There was a crooked man who had a crooked 1 euro coin. On lucky days it would flip the *heads* with probability $p = \frac{2}{3}$, and the *tails* with probability $p = \frac{1}{3}$, but on unlucky days it was the opposite ($p(\texttt{heads}) = \frac{1}{3}$, but $p(\texttt{tails}) = \frac{2}{3}$). There were equal probabilities of $\frac{1}{2}$ for lucky and unlucky days.
One morning he flipped the coin 5 times and altogether got three *heads* and two *tails*.
Let us introduce the following events:

- E (evidence): Five coin tosses result in three *heads* and two *tails*.
- H (hypothesis): The current day is lucky.

(A) Find $P(E|H)$ - the conditional probability of E given that the day is lucky.
(B) Find $P(E|H) \cdot P(H)$ - the probability that the day is lucky and E happens.
(C) Find $P(E|\overline{H})$ - the conditional probability of E given that the day is not lucky.
(D) Find $P(E|\overline{H}) \cdot P(\overline{H})$ - the probability that the day is unlucky and E happens.
(E) Find $P(E)$ - as the sum of two probabilities (E happened on a lucky day and also E happened on unlucky day).
(F) Find the conditional probability $P(H|E)$ - the likelyhood that the croocked man has a lucky day, given that the event E has happened.

Question 9.

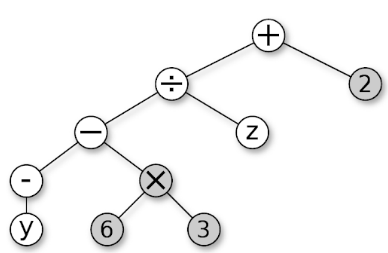

Figure 1. A tree for an expression.

The syntax tree describes an algebraic expression (please note the difference between the unary minus that flips the value of the variable y and the binary minus that subtracts the two subexpressions: $-y$ and 6×3).
(A) Write the preorder DFS traversal of this tree.
(B) Write the inorder DFS traversal of this tree.
(C) Write the postorder DFS traversal of this tree.
Note. In all 3 answers denote the unary minus with the tilde sign $\sim$, but the regular/binary minus with $-$.

Question 10.

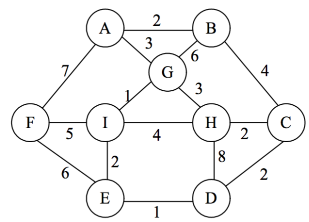

Figure 2. A graph with 9 vertices.

Run the Prim's algorithm on the weighted graph in Figure 2, start growing the tree from the vertex I.

Step	Newly Added Edge
Step 1	
Step 2	
Step 3	
Step 4	
Step 5	
Step 6	
Step 7	
Step 8	

What is the total weight of the obtained Minimum Spanning Tree?

Answers

Every problem is worth 15 points. The total for this final is 150 points.

Question 1.

(A) $X = A \cup B$ (Boolean OR means set union)
(B) $Y = (B \cap C) \cup \overline{A}$ (Boolean and means set intersection; negation means set complement)
(C) $|X| = |A| + |B| - |A \cap B| = 60 + 40 - 20 = 80$ (principle of inclusion-exclusion).
(D) $|Y|$ is all odd numbers and also four even numbers divisible by 15 $(30, 60, 90, 120)$. The total is $60 + 4 = 64$.
Grading.

- Each correct answer is 3 points (total 12).
- Explaining both **(C)** and **(D)** is another 3 points (total 3).
- Using wrong set notation ($\wedge$ instead of $\cap$ etc.) divides the number of points in half.
- 68 instead of 64 is 2 points (instead of 3).

Question 2.

In all the answers the largest and the smallest value are equal, because we know exactly how the two sets intersect; how many elements belong to just one of the sets A, B, and how many elements belong to the both sets.
(A) $|A \cup B| = |A| + |B| - |A \cap B| = 8 + 5 - 3 = 10$ (the principle of inclusion-exclusion).
(B) $|A \times (B \times B)| = 8 \cdot 5 \cdot 5 = 200$
(Cartesian product has size that is the product of all participant sets: one can combine three elements from the sets A, B and B in this many ways).
(C) $2^{2^3} = 2^8 = 256$ (the number of elements in the powerset of any set X can be obtained by raising 2 to the power $|X|$).
(D) $|A \oplus B| = (8 - 3) + (5 - 3) = 7$ (we remove the common elements from both A and B).
Grading.

- Each correct answer is 3 points (total 12).
- Expressions or verbal explanations of the answers is 3 points (total 3).

Question 3.

(A) $a_0 = 3$,
$a_1 = 4$,
$a_2 = 5 \cdot 4 - 6 \cdot 3 = 2$,
$a_3 = 5 \cdot 2 - 6 \cdot 4 = -14$,
$a_4 = 5 \cdot (-14) - 6 \cdot 2 = -82$,
$a_5 = 5 \cdot (-82) - 6 \cdot (-14) = -326$,
$a_6 = 5 \cdot (-326) - 6 \cdot (-82) = -1138$.
(B) The characteristic equation is obtained, if we try to find a_n in the form of a geometric progression r^n: $r^{n+2} = 5r^{n+1} - 6r^n$, or $r^2 - 5r + 6 = 0$. It has two roots: $r_1 = 2$, $r_2 = 3$.
(C) The general form of the expression for any iterative sequence b_n satisfying the relationship $b_{n+2} = 5b_{n+1} - 6b_n$ is as follows:

$$b_n = A \cdot 2^n + B \cdot 3^n,$$

where A, B are two constants that depend on the two initial values of the sequence b_n.
(D) We need to solve a system of two equations, to ensure that the formula $a_n = A \cdot 2^n + B \cdot 3^n$ has correct values for $n = 0$ and $n = 1$. We get the following system:

$$\begin{cases} A + B = 3, \\ 2A + 3B = 4. \end{cases}$$

Substitute $B = 3 - A$ into the second equation. We get that $2A + 9 - 3A = 4$ and $A = 5$. We also get that $B = -2$. Therefore the exact formula to calculate the sequence a_n is this:

$$a_n = 5 \cdot 2^n - 2 \cdot 3^n, \text{ where } n \geq 0.$$

This actually works, if we plug in the values calculated in **(A)** for $n = 0, \ldots, 6$.
Grading.

- Answer in **(A)** is 4 points.
- Answer in **(B)** is 4 points.
- Answer in **(C)** is 3 points.
- Answer in **(D)** is 4 points.

Question 4.

(A) The algorithm takes all numbers i from 1 to n^2 and divides them by all the smaller numbers $j < i$, and adds up all the obtained remainders.
(C) The outer loop is repeated n^2 times. The inner loop is repeated $1+2+3+\ldots+n^2$ times. This is an arithmetic progression. The sum of an arithmetic progression is the arithmetic mean of the first and the last member multiplied by the number of members:

$$f(n) = \frac{1 + n^2}{2} \cdot n^2 = \frac{n^4 + n^2}{2}.$$

(B) $f(n)$ is in $O(n^4)$. Therefore we can take $g(n) = n^4$. We can pick another $g(n)$ that is multiplied by some nonzero constant (such as $\frac{n^4}{2}$ or $17n^4$ or anything else - that also counts as a valid answer). Certainly, $f(n)$ is also in $O(n^k)$ for any $k > 4$, but the function $g(n) = n^4$ is the slowest growing.
Grading.

- Answer for **(A)** is 5 points.
- Answer for **(B)** (any 4th degree polynomial of n) is 5 points. An attempt to estimate some arithmetic progression with a different upper limit (say, $1 + 2 + \ldots + n$) gets 2 points.
- Answer for **(C)** is 5 points. If the answer is provided just for $n = 1000$ (not for any variable), then it is 4 points.

Question 5.
(A) If expressed as a product of two positive integers $120 = ab$, one of the divisors a or b would be smaller than $\sqrt{120} \approx 11$, and the other one would be bigger. We can easily list all the ways to express 120 as a product of two integers:

$$1 \cdot 120 = 2 \cdot 60 = 3 \cdot 40 = 4 \cdot 30 =$$

$$= 5 \cdot 24 = 6 \cdot 20 = 8 \cdot 15 = 10 \cdot 12,$$

and there are no other factorizations, since all the divisors less than 11 are already listed. Multiplying them all together would give

$$(120)^8 = 42998169600000000$$

(B) As a product of prime factors:

$$(120)^8 = (2^3 \cdot 3 \cdot 5)^8 = 2^{24} \cdot 3^8 \cdot 5^8.$$

Grading.

- Correct item **(A)** is 7 points (also floating point answers were fine - not all had easy access to the big integer arithmetic).
- Correct item **(B)** is 8 points.

Question 6.

Item	**Statement**	**True or False?**
(A)	R is reflexive	TRUE
(B)	R is symmetric	TRUE
(C)	R is antisymmetric	FALSE
(D)	R is transitive	TRUE
(E)	aRb iff $a = b$	FALSE

(A) Counterexample: None
(B) Counterexample: None
(C) Consider counterexample $a = 0$, $b = 11 \cdot 12 \cdot 13 = 1716$. While it is true that aRb and bRa, nevertheless $a \neq b$.
(D) Counterexample: None
(E) Counterexample is same as in **(C)**: $a = 0$, $b = 1716$.
Grading.

- One correct answer is 2 points (total 10).
- Counterexamples for **(C)** and **(E)** is 5 points (they are, in fact, the same).

Question 7. Answer: 17

- For 1 of 24 permutations $X = 4$ (all hats stay in place),
- For 0 permutations $X = 3$ (it is not possible for exactly three hats to stay in place, because then the 4th hat also returns to its owner),
- For 6 of 24 permutations $X = 2$ (there are $\binom{4}{2} = 6$ ways how to pick 2 hats that stay in place; and the remaining two hats can switch places only in one way),
- For 8 of 24 permutations $X = 1$ (there are $\binom{4}{1} = 4$ ways how to pick 1 hat that stays in place; and the remaining three hats can rotate in two ways).
- For the remaining $24 - (1 + 6 + 8) = 9$ permutations $X = 0$ (no hats stay in place).

Table 1
Random Variable X for the Hat Problem

Permutation	X	$X - E(X)$	$(X - E(X))^2$
ABCD	4	3	9
ABDC	2	1	1
ACBD	2	1	1
ACDB	1	0	0
ADBC	1	0	0
ADCD	2	1	1
BACD	2	1	1
BADC	0	−1	1
BCAD	1	0	0
BCDA	0	−1	1
BDAC	0	−1	1
BDCA	1	0	0
CABD	1	0	0
CADB	0	−1	1
CBAD	2	1	1
CBDA	1	0	0
CDAB	0	−1	1
CDBA	0	−1	1
DABC	0	−1	1
DACB	1	0	0
DBAC	1	0	0
DBCA	2	1	1
DCAB	0	−1	1
DCBA	0	−1	1
Mean	1	0	1

(A) $E(X) = \frac{1}{24} \cdot 4 + \frac{6}{24} \cdot 2 + \frac{8}{24} \cdot 1 = 1$. This means that the expected number of hats that stay in place is exactly 1.
(B) For all 24 permutations, subtract the value $E(X) = 1$ from every hat experiment outcome. Define $x_1, \ldots, x_{24}$ - all 24 values of the random variable X (exactly one value is 4, exactly six values are 2, exactly 8 values are 1, exactly 9 values are 0):

$$V(X) = \frac{\sum_{i=1}^{24} (x_i - 1)^2}{24} = \frac{24}{24} = 1.$$

Therefore, $V(X) = 1$ (variance also equals 1, but the unit of measurement is not hats but "hats squared").
Note. $E(X) = 1$ is the arithmetic mean over the column X, but $V(X) = 1$ is the arithmetic mean over the column $(X - E(X))^2$ (see Table 1).
Grading (theoretically max=20, but most results are not that high).

- Correctly found $E(X)$ is 5 point.

- Correctly found $V(X)$ is 5 point.
- Justified computation for $E(X)$ is 5 points.
- Justified computation for $V(X)$ is 5 points.

Question 8.

(A) $P(E|H)$ is the outcome of the Binomial distribution: There are $n = 5$ coin-toss experiments; the probability of success for any single experiment is $p = \frac{2}{3}$ (since we know that the day is lucky and hypothesis H holds). Therefore,

$$P(E|H) = \binom{5}{3} p^3 (1-p)^2 = 10 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^2 = \frac{80}{243}$$

(B) $P(E|H) \cdot P(H) = \frac{80}{243} \cdot \frac{1}{2} = \frac{40}{243}$, since $P(H) = \frac{1}{2}$ (the *a priori* probability of a lucky day is exactly $1/2$).
(C) $P(E|\overline{H})$ is the outcome of the Binomial distribution: Again, there are $n = 5$ coin-toss experiments, but now the probability of a single experiment is just $p = \frac{1}{3}$. Therefore,

$$P(E|\overline{H}) = \binom{5}{3} p^3 (1-p)^2 = 10 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^2 = \frac{40}{243}$$

(D) $P(E|\overline{H}) \cdot P(\overline{H}) = \frac{40}{243} \cdot \frac{1}{2} = \frac{20}{243}$.
(E) We can compute $P(E)$ as the sum of two mutually incompatible events: event E can happen either on a lucky day or on an unlucky day:

$$P(E) = P(E|H) \cdot P(H) + P(E|\overline{H}) \cdot P(\overline{H}) = \frac{40}{243} + \frac{20}{243} = \frac{60}{243}.$$

(F) Use Bayes formula:

$$P(H|E) = \frac{P(E|H) \cdot P(H)}{P(E|H) \cdot P(H) + P(E|\overline{H}) \cdot P(\overline{H})} = $$
$$= \frac{P(E|H) \cdot P(H)}{P(E)} = \frac{\frac{40}{243}}{\frac{60}{243}} = \frac{2}{3}.$$

Bayes formula is intuitive: It shows the proportion of the subcase $P(E|H) \cdot P(H)$ (i.e. event E hapens on a lucky day) out of the whole probability $P(E) = P(E|H) \cdot P(H) + P(E|\overline{H}) \cdot P(\overline{H})$ (i.e. event E happens either on a lucky or unlucky day).
Grading (theoretically max=20, but most results are not that high).

- Any item from **(A)** to **(E)** is 3 points.
- Bayes formula or a similar expression finding the reverse conditional probability in **(F)** in 5 points.

Question 9.
(A) `+ : − ∼ y × 6 3 z 2`,
(B) `y ∼ − 6 × 3 : z + 2`,
(C) `y ∼ 6 3 × − z : 2 +`.
Note. In inorder traversal **(B)** we first visit the first subtree (e.g., `y`), and only then the parent node (e.g., unary minus ∼). See (Rosen2019, p.811).
Grading.

- Each correctly written expression is
- Item **(B)** with switched order of the unary minus and its child node y is 3 points instead of 5.
- Minor typos in single characters get 4 or 5 points.
- Any major differences from the correct result do not get points.

Question 10.

We start from vertex I. At every step we grow the tree by a single edge (so that it stays connected and the newly added edge has the smallest possible weight).

Step	Newly Added Edge
Step 1	IG, $w = 1$
Step 2	IE, $w = 2$
Step 3	ED, $w = 1$
Step 4	DC, $w = 2$
Step 5	CH, $w = 2$
Step 6	GA, $w = 3$
Step 7	AB, $w = 2$
Step 8	IF, $w = 5$

The total weight of all added edges (same as the total weight of the MST) is 18.

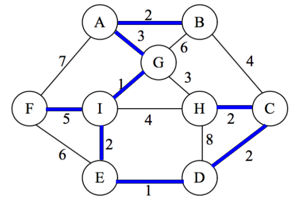

Figure 3. MST edges shown in blue.

Grading.

- Incorrectly adding up weights could subtract 1 or 2 points from the total.
- Adding 9 edge weights (or any other number instead of 8 weights) and getting incorrect sum is 11 points (instead of 15).
- Not showing the edges in answers (or displaying them in an order that differs from Prim's algorithm), but still getting something close to MST is about 8 points.