Midterm Preparation: Sample Problems

Midterm may contain 3-5 computation problems – Part A, and also 1-2 analysis problems – Part B and/or 1-2 proofs (see Part C). Expect around 8 problems in the midterm (it would be about 15 minutes per problem).

This document is **not** a mock exam; it contains 42 sample questions (many more than you would expect to get in actual midterm). This material is meant for discussion in order to prepare and to show some variations of the problems.

Proofs will closely follow some pattern that has been shown in a homework exercise (a variant of an existing proof). Also the computation and analysis tasks will follow a recognizable pattern that is presented in this preparation material or any of the quizzes or homeworks.

Besides the proofs, midterm can contain procedural problems (applying algorithms in one particular situation); analysis tasks (for example, formalize something with predicates or quantifiers or apply some other conceptual knowledge) and so on. Even, if you are following a known procedure you need to add comments to explain what you are doing. They can earn you partial credit, even if your result turns out incomplete.

Part A. Computational Problems

In this part we apply known algorithms to particular situations. The goal is to obtain correct result and and add a short explanation, what you did and why.

A1. Propositional Logic. Question 1. Consider this Boolean expression: $E = p \rightarrow q \rightarrow r$. Implication ($\rightarrow$) is right-associative.

(a) Write an equivalent Boolean expression using only conjunctions ($\wedge$) and negations ($\neg$).

(b) Fill in the missing parts in the truth table:

p	q	r	E
T	T	T	...
T	T	F	...
T	F	T	...
T	F	F	...
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

Question 2. Assume the following precedence order and associativity of the 5 Boolean operations:

`~a`	1st precedence	right-associative
`a/\b`	2nd precedence	right-associative
`a\/b`	3rd precedence	right-associative
`a<->b`	4th precedence	left-associative
`a->b`	5th precedence	right-associative

Show step-by-step how would you restore parentheses in the following Boolean expression (each step adds one pair of parentheses – in the order that follows from the rules of precedence and associativity):

```
a -> b /\ ~ c \/ ~ ~ d <-> e -> f <-> g
```

Question 3. The diagram on Figure 1 shows different options how 5 people – Bill, John, Jim, Simon and Mike – can work in a yard. (Variable `Bill` is true iff Bill works in the yard, and the whole expression evaluates to true, iff it shows a possible combination how a team of boys can work in the yard).

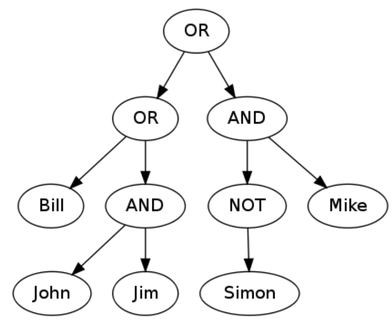

Figure 1. Boolean expression as a tree.

(A) Write this tree as a Boolean expression using Boolean connectives ($\neg$, $\wedge$, etc.).

(B) Write the same expression and ensure that it does not use any unnecessary parentheses (precedence and associativity is same as in Question 2).

(C) Write an equivalent expression using just two Boolean connectives (negation $\neg$ and disjunction $\vee$).

A2. Sets and Quantifiers. Question 4. The universe U is all integers between 1 and 600. $K_2 \subseteq U$ denotes all even numbers from U. Similarly, $K_3 \subseteq U$ denotes all numbers divisible by 3; $K_5 \subseteq U$ denotes all numbers divisible by 5.

(A) Draw Euler-Venn diagram with a big rectangle (the set U), the subsets K_2, K_3, K_5 as three intersecting circles.

(B) Shade the region that corresponds to the set $(K_2 \cap \overline{K}_3) \cup K_5$.

(C) Find the cardinality of the set $\left|(K_2 \cap \overline{K}_3) \cup K_5\right|$, justify your answer.

(D) Describe, which elements are in the set $(K_2 \cap \overline{K}_3) \cup K_5$ in English. For example: *"All ... such that ... is divisible ... and/or is not divisible..."*.

Question 5. In Figure 2 a red square on the intersection of row i and column j means that the predicate $L(i, j)$ is true (person i loves person j). White square means that the predicate $L(i, j)$ is false. Here $i, j \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$. Which statement is shown in the left picture?

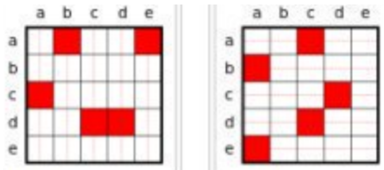

Figure 2. Predicate on a Cartesian product.

(A) Everyone is loved by someone.

(B) Eveyone loves someone.

(C) Someone loves everyone.

(D) Someone is loved by everyone.

Which statement is shown in the right picture?

(A) $\forall x\, \exists y,\ L(y, x)$.

(B) $\forall x\, \exists y,\ L(x, y)$.

(C) $\exists x\, \forall y,\ L(x, y)$.

(D) $\exists x\, \forall y,\ L(y, x)$.

A3. Algorithms and Big-O Notation. In these problems you can verify the definition of the Big-O Notation or find the limit $f(n)/g(n)$ as $n \to \infty$.

Question 6. For each function find the smallest k such that $f(n)$ is in $O(n^k)$. Justify your answer.

(A) $f(n) = \sum_{j=1}^{n} (j^3 + j \log_2 j)$.

(B) $f(n) = n^3 + \sin n^7$.

(C) $f(n) = 1^2 + 2^2 + \ldots + n^2$.

Question 7. We say that the functions $f(n)$ and $g(n)$ *are of the same order*, if $f(n)$ is in $O(g(n))$ and $g(n)$ is in $O(f(n))$. Find all pairs of functions in this list that are of the same order:

$$n^2 + \log_2 n,\ 2^n + 3^n,\ 100n^3 + n^2,\ n^2 + 2^n,$$

$$n^2 + n^3,\ 3n^3 + 2^n.$$

A4. Number Theory. Question 8.

(A) Somebody has written a long hexadecimal number `F0F0F0...F0F` – it uses 28 digits "F" separated by 27 digits "0". Express the value of this number as a short expression (without ...), using the formula of a geometric progression: $\frac{b_1(q^{n+1}-1)}{q-1}$.

(B) Somebody has written an infinite hexadecimal fraction `0.F0F0F0....` Express it as a rational number in decimal notation.

Question 9.

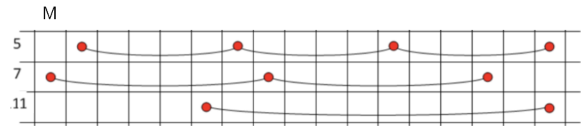

Figure 3. 3 arithmetic progressions.

(A) Find any number $M \in \mathbb{Z}^+$ that gives the following remainders when divided by 5, 7 and 11 (to ensure that in Figure 3 red circles denote multiples of these numbers).

$$\begin{cases} M \equiv 4 \pmod{5} \\ M \equiv 0 \pmod{7} \\ M \equiv 6 \pmod{11} \end{cases}$$

(B) Find the arithmetic progression containing all such numbers M.

Question 10.

(A) Alice has 13-cent coins; Bob has 21-cent coins. Alice wants to pay Bob exactly 1 cent. How to do this?

(B) Alice has 21-cent coins; Bob has 13-cent coins. Alice wants to pay Bob exactly 1 cent. How to do this?

(C) Solve the congruence equation – find x such that $13x \equiv 1\ (mod\ 21)$.

(D) Solve the congruence equation – find x such that $13x \equiv 4\ (mod\ 21)$.

Question 11.

(A) Write all positive powers of number 2 $(2^1, 2^2, 2^3, \ldots)$ modulo 11 until you find a loop: i.e. some remainder repeats itself. How long is the period?

(B) Write all negative powers of number 2 modulo 13 $(2^{-1}, 2^{-2}, 2^{-3}, \ldots)$ until you find a loop. How long is the period?

Note. A negative power, for example 2^{-k} is such that $2^{-k} \cdot 2^k \equiv 1 \pmod{11}$.

Part B. Analysis Tasks

In these problems you have to apply concepts (such as predicates or quantifiers) to new situations, describe algorithms or procedures for new tasks or sort cases in adaptive ways.

B1. Propositional Logic. **Question 12.** Among the people A, B, C one is a truth-teller, the other two are liars. Every person (A, B, and C) has a closed box in front of himself/herself. Exactly one of the boxes has a candy inside. A, B, C know everything about each other and the location of candy.
Find out, which YES/NO questions you need to ask to find out, which box contains candy. Ask as few questions as possible ("brute-force" strategies that ask clearly redundant questions may only get partial credit).

Question 13.

(a) Find out, if this formula is a tautology:

$$((\mathtt{A} \vee \mathtt{B}) \wedge (\mathtt{A} \rightarrow \mathtt{C}) \wedge (\mathtt{B} \rightarrow \mathtt{C})) \rightarrow \mathtt{C}.$$

(b) Find out, if this formula is satisfiable:

$$\neg(((\mathtt{A} \vee \mathtt{B}) \wedge (\mathtt{A} \rightarrow \mathtt{C}) \wedge (\mathtt{B} \rightarrow \mathtt{C})) \rightarrow \mathtt{C}).$$

Note. A Boolean expression is called *tautology*, if it is true for all possible truth values of its variables. An expression is called *satisfiable*, if there is a way to assign variables so that it becomes true.

Question 14. Let us have three people – Ada, Barbara and Cecilia. Every day some of them are charging batteries for Lego robots. Let us have propositional variables a, b, c (a variable a, b, c is true, iff today batteries were charged by Ada, Barbara or Cecilia respectively). Write a Boolean expression telling that today batteries were charged by **exactly** two people out of the three. You can use variables a, b, c and all 5 Boolean connectives ($\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$).

B2. Sets and Quantifiers. **Question 15.** We define the set of all bounded functions defined on the interval $(a; b)$ as functions that have all their values between two bounds: there is a positive number M such that all all values of the function f are in the interval $[-M; M]$. With predicates and quantifiers:

$$\exists M \in \mathbb{R}^+,\ \forall x \in \mathbb{R}\ \ x \in (a; b) \rightarrow |f(x)| \leq M.$$

In Figure 4 one of the functions (the red one) is bounded, the other one is unbounded.

(A) Write an expression with predicates and quantifiers to express the statement: "Function $f : (a; b) \rightarrow \mathbb{R}$ is **not** bounded."

(B) Find an example of a function that is defined and bounded in the interval $(0; 1)$.

(C) Find an example of a function that is defined, but unbounded in the interval $(0; 1)$.

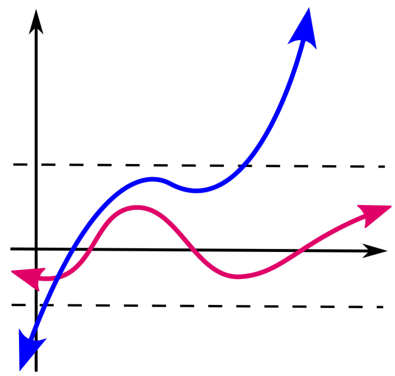

Figure 4. A bounded function.

(D) What functions are described by another predicate logic expression, where we switch the order of quantifiers:

$$\forall x \in \mathbb{R}\ \exists M \in \mathbb{R}^+,\ x \in (a; b) \rightarrow |f(x)| \leq M.$$

Question 16. An infinite sequence $a_0, a_1, a_2, \ldots$ is called *purely periodic*, if all the members of it repeat the same finite pattern over and over again. For example, $3, 3, 3, 3, \ldots$ or $1, 3, 1, 3, 1, 3, \ldots$ are purely periodic. Meanwhile, $1, 6, 6, 6, 6, \ldots$ or $1, 4, 1, 4, 2, 1, 3, 5, \ldots$ (the digits of $\sqrt{2}$) are not purely periodic (they are either "eventually periodic" with some digits preceding the period or not periodic at all).

(A) Write an expression to tell that a given sequence $(a_n)_{n \in \mathbb{N}}$ is purely periodic.

(B) Write an expression to tell that a given sequence $(a_n)_{n \in \mathbb{N}}$ is **NOT** purely periodic.

Question 17. Consider the set $A = \{1, 2, 3, 4, 6, 12\}$. We define a predicate

$$P : A \times A \rightarrow \{\mathtt{True}, \mathtt{False}\},$$

where $P(i, j)$ is true iff the divisibility by i implies divisibility by j (for any $i, j \in A$). For example $P(12, 2) = \mathtt{True}$, since any number that is divisible by $i = 12$ is also divisible by $j = 2$.

(A) Is the 2-argument predicate P *reflexive* – does it satisfy the logic formula:

$$\forall i \in A,\ P(i, i).$$

(B) Is the predicate P *symmetric* – does it satisfy the logic formula:

$$\forall i \in A,\ \forall j \in A,\ P(i, j) \rightarrow P(j, i).$$

(C) Is the predicate P *transitive* – does it satisfy the logic formula:

$$\forall i, j, k \in A,\ P(i, j) \land P(j, k) \to P(i, k).$$

(D) Is the predicate P *connex* – does it satisfy the logic formula:

$$\forall i, j \in A,\ P(i, j) \lor P(j, i).$$

Note 1. Even though the predicate formulas can be verified by checking all possible combinations of $i, j, k \in A$ (the set A is finite), it is much faster to reason by the properties of divisibility.
Note 2. For each statement either add a short justification or provide a counterexample.
B3. Algorithms and Big-O Notation. Question 18. Let A, B be subsets from the same universe $U = \{1, 2, \ldots, n\}$. We want to find the intersection of these two sets $A \cap B$.

(A) Describe an algorithm (using steps written in English or some pseudocode) to find and to output all numbers that are in this intersection $A \cap B$. (You can call the algorithms that are in the K.Rosen's textbook - for sorting, linear or binary search and similar. But write out your assumptions, so that it is unambiguous, which method you are using.)

(B) Estimate the time complexity $O(f(n))$: provide a function that estimates the worst-case time required as the size of the universe U grows.

Question 19.

1. Prove that $f(n) = n^2 + n \cdot \ln n$ is in $O(n^2)$ (i.e. $g(n) = n^2$) by finding the limit of the ratio:

$$\lim_{n\to\infty} \frac{n^2 + n \cdot \ln n}{n^2}.$$

   Or by directly checking the definition of the Big-O notation (finding the two constants C and k that give you the estimate: $n > k \to |f(n)| \leq C|g(n)|$.

2. Is the function $g(n) = n^2$ also in $O(f(n))$, where $f(n) = n^2+n\cdot\ln n$? Once again, justify by finding a limit or by checking the definition.

B4. Number Theory. Question 20. Let us consider the following statement:

Bezout's identity: Let a and b be integers and d is their greatest common divisor: $\gcd(a, b) = d$. Then, there exist integers x and y such that $ax+by = d$.

(A) Let us have parameters $a, b \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$ such that $\gcd(a, b) = d$. Write the Bezout's identity using predicates and quantifiers (you can also use equality and four arithmetic operations).

(B) Using the same notation as before, write another statement using predicates and quantifiers: "The modular equation $ax \equiv c \pmod b$ has a solution if and only if c is a multiple of d."

(C) Using the same notation as before, write another statement using predicates and quantifiers: "The greatest common divisor d is the smallest positive integer that can be expressed as $ax + by$ for the given a, b and any integers x, y."

Note 1. In this expression a, b, c, d are *free variables*; they are given and can be used in your formulas. You can also use any number of *bound variables*.
Note 2. We call c a *multiple* of d iff c can be expressed as some integer number times d.

Question 21. Somebody has proposed a way to encode finite sequences of natural numbers $\mathbb{N}$ (i.e. non-negative integers: $\mathbb{N} = 0, 1, 2, \ldots$) as natural numbers. To encode a sequence $(a_1, a_2, \ldots, a_k)$ we take the first k primes ($p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on) and build a number:

$$f(a_1, a_2, \ldots, a_k) = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots$$

For example, the sequence $(1, 0, 2)$ is encoded as $2^1 \cdot 3^0 \cdot 5^2 = 50$, but the sequence $(1, 1, 1, 1)$ is encoded as $2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 = 210$.

1. Is the function $f : \mathbb{N}^* \to \mathbb{N}$ a surjection?
2. Is the function $f : \mathbb{N}^* \to \mathbb{N}$ an injection?
3. Can you suggest a bijective function $g : \mathbb{N}^* \to \mathbb{N}$ to encode any finite sequence of natural numbers with a single natural number?

Note. By $\mathbb{N}^*$ we denote a list of all finite tuples:

$$\mathbb{N}^* = \{()\} \cup (\mathbb{N}) \cup (\mathbb{N} \times \mathbb{N}) \cup \ldots.$$

It contains one list of length 0,
It contains $(\mathbb{N})$ all single-element lists (0), (1), etc.,
It contains $(\mathbb{N} \times \mathbb{N})$, i.e. all two-element pairs $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$, etc.
And so on.

Question 22. Consider the set of all square functions ($f : \mathbb{R} \to \mathbb{R}$) that can be expressed as $f(x) = ax^2+bx+c$ with integer coefficients a, b, c. Is the set of all such square functions finite? countable? uncountable? Justify your answer.

Part C. Proofs

In proof problems the goal is to prove some general statement by showing every essential step of your reasoning. Below we describe various proof strategies and give some sample statements that can be proven by that strategy.

Translate into Algebra
In K.Rosen's textbook these are called "direct proofs" of IF-THEN statements. You assume that the condition is true, introduce some notation and prove that also the conclusion is true. You sometimes need to sort cases.

Question 23. Prove that, if n is odd, then n^2 is also odd.

Question 24. Prove that, if the decimal notation of a number n ends with the digit "5", then n^2 ends with digits "25".

Question 25. Prove that $n^2 \equiv 5 \pmod{11}$ is true iff $n \equiv 4 \pmod{11}$ or $n \equiv -4 \pmod{11}$.

Proofs by Contradiction
In these examples you assume that the statement is false and get a contradiction.

Question 26. Prove that there are infinitely many primes.

Question 27. Prove that there are infinitely many primes of the form $4n + 3$.

Question 28. Prove that there are infinitely many primes that divide some value of the polynomial $P(n) = n^2 + n + 1$.

Question 29. Prove that $\sqrt{2}$ is irrational.

Question 30. $\log_2 10$ is irrational.

Building Bijective Functions (Countable)
Any combinations of these tactics can be used to prove that two sets have the same cardinality (and that there is a bijective function from one set to another).

Question 31. There is a bijection from $\mathbb{Z}^+ \cup \{1, 2, \ldots, n\}$ to $\mathbb{Z}^+$.
Hint. One can add n new guests to the Hilbert's hotel.

Question 32. There is a bijection from $\mathbb{Z}^+ \times \{1, 2\}$ to $\mathbb{Z}^+$.
Hint. If there are two infinite buses with guests: $(1, 1), (2, 1), (3, 1), \ldots$ and $(1, 2), (2, 2), (2, 3), \ldots$, then they can be hosted in a single Hilbert's hotel.

Question 33. There is a bijection from $\mathbb{Z}^+ \times \mathbb{Z}^+$ to $\mathbb{Z}^+$.
Hint. Show how infinitely many infinite buses can be hosted in a single Hilbert's hotel (similar to Question 9 from HW1).

Question 34. There is a bijection from $\mathbb{Q}$ to $\mathbb{Z}^+$.
Hint. Same as above, but now you need to combine three sets of numbers into the same Hilberts hotel (positive rationals, negative rationals and 0).

Building Bijective Functions (Uncountable)

Question 35. There is a bijective function from any closed segment $[a; b]$ to any other closed segment $[c; d]$ (regardless of their lengths).
Hint. This can be achieved by a simple linear function.

Question 36. There is a bijective function from any open segment $(a; b)$ to any other open segment $(c; d)$.
Hint. This also can be achieved by a linear function.

Question 37. There is a bijective function from any open segment $(a; b)$ to the set of all real numbers $\mathbb{R}$ or to the half-line of all positive reals $(0; +\infty)$.
Hint. You can use a continuous function with one or two vertical asymptotes. Such as $f(x) = x/(1 - x)$ or $f(x) = \tan x$.

Question 38. There is a bijective function from a semi-open segment $[0; 1)$ to an open segment $(0; 1)$.
Hint. Map the extra point (argument $x = 0$) to any point inside the target interval $(0; 1)$, then change values for an infinite sequence (bounded inside $(0; 1)$) in the same way as Hibert's hotel.

Proving there is no Bijection
All these proofs use Cantor's diagonalization. (See subsection "2.5.3 An uncountable set" in the textbook; pp. 183-187.)

Question 39. Numbers on $(0; 1)$ are uncountable.

Question 40. All real numbers $\mathbb{R}$ are uncountable - there is no bijection from $\mathbb{R}$ to $\mathbb{Z}^+$ or to $\mathbb{Q}$.

Question 41. All infinite sequences of integers are uncountable (Also, all non-decreasing sequences of integers are uncountable.

Question 42. The Power-set of any countable set is uncountable. For example, there is no bijection from $\mathcal{P}(\mathbb{Z}^+)$ to $\mathbb{Z}^+$ (i.e. no mapping from the set of all subsets of positive integers to the set of integers themselves).