Quiz 4: Sets

Question 1. Define the universe U to be all possible remainders when we divide by 360: $\{0, 1, 2, \ldots, 359\}$. Also define 3 subsets in this universe:

$$\begin{cases} K_2 &=& \{x \in U \mid x \text{ divisible by } 2\}, \\ K_3 &=& \{x \in U \mid x \text{ divisible by } 3\}, \\ K_5 &=& \{x \in U \mid x \text{ divisible by } 5\}, \end{cases}$$

Denote by Φ the subset of U containing all numbers that are mutually prime with 360 (no common divisors greater than 1): $\Phi = \{1, 7, 11, 13, \ldots, 359\}$. Which set equality is valid regarding the subset Φ:
(A) $\Phi = (K_2 \cup K_3 \cup K_5)$
(B) $\Phi = (K_2 \cap K_3 \cap K_5)$
(C) $\Phi = \left(\overline{K_2} \cup \overline{K_3} \cup \overline{K_5}\right)$
(D) $\Phi = \left(\overline{K_2} \cap \overline{K_3} \cap \overline{K_5}\right)$
(E) $\Phi = \left(\overline{K_2 \cap K_3} \cup \overline{K_2 \cap K_5} \cup \overline{K_3 \cap K_5}\right)$
Pick your answer as a single letter like this: `G`

Question 2. Find the size of the set you constructed in the previous example.
Write your answer as a single non-negative integer like this: `17`

Question 3. We have the following sets:
A is the set of all finite sequences of even positive positive numbers (such as $(6, 22, 10, 14, 2, 6)$, and so on)
B is the set of all infinite nondecreasing lists of even positive numbers (such as $(40 \leq 40 \leq 42 \leq 46 \leq \ldots)$, and so on)
C is the set of all infinite nonincreasing lists of even positive numbers (such as $(64 \geq 58 \geq 58 \geq 54 \geq \ldots)$, and so on).
Clearly, all three sets are infinite. Determine their cardinalities - which list of cardinalities is equal to the list $(|A|, |B|, |C|)$?
(A) $(|\mathbb{N}|, |\mathbb{N}|, |\mathbb{N}|)$. **(B)** $(|\mathbb{N}|, |\mathbb{R}|, |\mathbb{N}|)$. **(C)** $(|\mathbb{N}|, |\mathbb{N}|, |\mathbb{R}|)$. **(D)** $(|\mathbb{N}|, |\mathbb{R}|, |\mathbb{R}|)$. **(E)** $(|\mathbb{R}|, |\mathbb{R}|, |\mathbb{R}|)$.
Pick your answer as a single letter like this: `G`

Question 4. Let $f(x) = (x^2)$ **mod** 11. Find the set $f(S)$ if $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
Write the list of elements of $f(S)$ as a sorted list like this: `1,2,3`

Question 5. How many 2-element sets are there in the powerset $\mathcal{P}(\{\{\mathtt{A}, \mathtt{B}\}, \mathtt{C}, \mathtt{D}, \mathtt{E}\})$?
Write your answer as a non-negative integer like this: `17`

Question 6. Given two sets $A = \{x, y\}$ and $B = \{x, \{x\}\}$, check, if statements are true or false:
(A) $x \subseteq B$.
(B) $\emptyset \in \mathcal{P}(B)$.
(C) $\{x\} \subseteq A - B$.
(D) $|\mathcal{P}(A)| = 4$.
Write your answer as a sorted list of letters (which are true) like this: `A,B,C,D`

Question 7. We define functions $g : A \rightarrow A$ and $f : A \rightarrow A$, where $A\{1, 2, 3, 4\}$ by listing all argument-value pairs: $f = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$, $g = \{(1, 3), (2, 1), (3, 4), (4, 2)\}$. Find the value pairs for the function $(f \circ g)^{-1}$.
Write your answer as a comma-separated list like this: `(1,1),(2,2),(3,3),(4,4)`

Question 8. Find the value of this infinite sum: $1 - 1/3 + 1/9 - 1/27 + 1/81 - \ldots$.
Write your answer as a simple fraction: `P/Q`

Question 9. It is known that the function $f(n) = n^3 + 88n^2 + 3$ is in $O(n^3)$ – its asymptotic growth is as fast as the growth of the function $g(n) = n^3$. $\exists C \in \mathbb{Z}^+ \; \exists n_0 \in \mathbb{Z}^+ \; \forall n \in \mathbb{Z}^+$,
$(n > n_0 \rightarrow |f(n)| \leq C \cdot |g(n)|)$ Find the smallest positive integer C that would satisfy the above definition, and for your C find the smallest possible n_0.
Write your answer (C, n_0) as a pair of two numbers like this: `17,17`

Question 10. "Big O notation" allows to arrange functions according to the their growth rate for large n. Identify, which list of functions is such that the first element of this list is in the big-O of the next element of that list and so on. (Intuitively, the first element in the list is the slowest growing function, the last element is the fastest growing one.)

(1) $\log(n^{10})$, **(2)** $(\log n)^2$, **(3)** $\log\log n$, **(4)** $n \log n$, **(5)** $\log(n!)$, **(6)** $\log 2^n$.

Write your answer as a comma-separated list like this: `1,2,3,4,5,6`

Question 11. Digits of all rational numbers P/Q in $(0;1)$ are eventually periodic: they infinitely repeat some group of digits (the period) starting from some place. For example, the fraction $11/205 = 0.05(36585)$ has period of 5 digits and a pre-period "05" of just two digits. Find the predicate logic expression that tells that sequence of digits $d(1), d(2), d(3), \ldots$ is eventually periodic (it may have pre-period of any length, including length zero).

(A) $\exists N \in \mathbb{Z}^+\ \exists T \in \mathbb{Z}^+\ \forall n \in \mathbb{Z}^+$,
$(n \geq N - 1 \rightarrow d(n) = d(n+T))$.

(B) $\exists N \in \mathbb{Z}^+\ \forall n \in \mathbb{Z}^+\ \exists T \in \mathbb{Z}^+$,
$(n \geq N - 1 \rightarrow d(n) = d(n+T))$.

(C) $\forall n \in \mathbb{Z}^+\ \exists N \in \mathbb{Z}^+\ \exists T \in \mathbb{Z}^+$,
$(n \geq N - 1 \rightarrow d(n) = d(n+T))$.

(D) $\forall n \in \mathbb{Z}^+\ \forall N \in \mathbb{Z}^+\ \exists T \in \mathbb{Z}^+$,
$(n \geq N - 1 \rightarrow d(n) = d(n+T))$.

Pick your answer as a single letter like this: `G`

Answers

Question 1. Answer **(D)**.
Any number that is mutual prime with $360 = 2^3 \cdot 3^2 \cdot 5$ is not divisible by any of the primes $2, 3, 5$. And also vice versa. This set is expressed as intersection of the complements:
$\Phi = \left(\overline{K_2} \cap \overline{K_3} \cap \overline{K_5}\right)$

Question 2. Answer: `96`.

$$|\Phi| = 360 \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) \cdot \left(1 - \frac{1}{5}\right) =$$
$$= 360 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 96.$$

In the above formula we start with all 360 elements; then we throw out one half (all that are divisible by 2); then from the remaining ones we throw out one third (all that are divisible by 3); finally from the remaining numbers we throw out one fifth (all that are divisible by 5). Since divisibility by 2 does not affect divisibility by 3 and 5 (they are independent), all the ratios can be multiplied.

Another solution: Since we know the sizes of each set of numbers divisible by $2, 3, 5$:

$$|K_2| = 180,\ |K_3| = 120,\ |K_5| = 72.$$

We can express their union by *inclusion-exclusion principle*:

$$|K_2 \cup K_3 \cup K_5| \ = \ |K_2| \ + \ |K_3| \ + \ |K_5| \ -$$
$$- \ |K_2 \cap K_3| \ - \ |K_2 \cap K_5| \ - \ |K_3 \cap K_5| \ + \ |K_2 \cap K_3 \cap K_5| \ =$$
$$= \ 180 + 120 + 72 - 60 - 36 - 24 + 12 = 264.$$

We then apply De Morgan's law to find the count of all elements that are *outside* that union of $K_2 \cup K_3 \cup K_5$:

$$\left|\overline{K_2} \cap \overline{K_3} \cap \overline{K_5}\right| = \left|\overline{K_2 \cup K_3 \cup K_5}\right| = 360 - 264 = 96.$$

Question 3. Answer: `B`.

- A (the set of all finite sequences of even natural numbers can be enumerated with numbers from $\mathbb{N}$). You can encode every such sequence in a finite alphabet of 13 symbols, using just digits, commas and parentheses. For example, `(6, 22, 10, 14, 2, 6)`. The shortest encoding is `(1)` –it consists of just three symbols: two parentheses and a digit. There can be only finite number of such lists of length 3; we sort the all lexicographically (i.e. in some alphabetical order), and assign them numbers.
  After that we enumerate all lists writeable with 4 symbols (sorted lexicographically) and so on. Eventually all the sequences will be sorted.
- B (the set of all infinite nondecreasing sequences of even numbers) has cardinality $\mathbb{R}$. You can repeat the diagonalization argument: Assume from the contrary that the elements from B can be enumerated: we get infinitely many infinite sequences $b_1, b_2, \ldots$. Then take the first element from b_1 (and pick some even number that is bigger than that); then take the second element from b_2 (and pick some even number that is bigger than that; plus it is bigger than all the previously picked numbers), and so on.
  You can also encode any subset $A \subseteq \mathbb{N}$ as such sequence (simply arrange all the elements in increasing order and multiply them by 2 to get even numbers). We get that B has at least as many elements as $\mathcal{P}(\mathbb{N})$.
- C (the set of all nonincreasing infinite sequences can be enumerated). Since the sequence is non-increasing, it can have only finitely many places where it actually decreases; since natural numbers cannot decrease infinitely. We can encode all the "constant runs" of the sequence as pairs:

  $$(64, 58, 58, 54, 50, 50, 50, 50, 2, 2, \ldots) \rightarrow$$
  $$\rightarrow ((64, 1), (58, 2), (54, 1), (50, 4), (2, \infty)).$$

  As we saw before, all the finite sequences that are encoded in an alfabet of 14 symbols (10 digits, 2 parentheses, commas and infinity) can be enumerated.

Question 4. Answer: `0,1,3,4,5,9`.
We can square each number, compute the remainder and sort the results (and eliminate duplicates).

Question 5. Answer: `6`.
The set $\{\{\mathtt{A},\mathtt{B}\},\mathtt{C},\mathtt{D},\mathtt{E}\}$ has 4 elements (`A`, `B` are always glued together). There are 6 ways to select two out of four elements. (Can be computed as a binomial coefficient $C_4^2 = \frac{4!}{2!2!}$ or simply by listing all the 6 pairs.

Question 6. Answer: `B,D`.
x cannot be a subset of A (since it is not a set itself). $\{x\}$ is not a subset of $A - B = \{y\}$.

Question 7.
Answer: `(1,3),(2,2),(3,4),(4,1)`.
We first compute $f \circ g$ (to get $(f \circ g)(x) = f(g(x))$ we first apply g, then f):

$$f \circ g = \{(1,4),(2,2),(3,1),(4,3)\}.$$

The inverse happens, if we switch the order in all these pairs (4 maps back to 1 etc.)

$$(f \circ g)^{-1} = \{(1,3),(2,2),(3,4),(4,1)\}.$$

Question 8. Answer: `3/4`.
The sum of the infinite geometrical progression is $b_1/(1-q)$. In our case:

$$\frac{1}{1-(-1/3)} = \frac{1}{4/3} = \frac{3}{4}.$$

Question 9. Answer: `2,88`. Clearly, $f(n) = |n^3 + 88n^2 + 3|$ cannot be smaller than $C \cdot |n^3|$, if $C = 1$, because $88n^2$ is always positive and makes $f(n)$ larger than simply n^3.
If we take $C = 2$, then the inequality starts to hold for all $n > 88$. It is possible to prove that for such n:

$$n^3 + 88n^2 + 3 = n^2(n+88) + 3 =$$

$$= n^2(n+n) + n^2(88-n) + 3 = 2n^3 + n^2(88-n) + 3 \leq 2n^3.$$

The last inequality is true, since $n^2(88-n) + 3 < 0$ for any $n > 88$.

Question 10. Answer: `3,1,2,6,4,5` (or `3,1,2,6,5,4`).
Logarithm of a logarithm is a very slowly growing function; $\log n^{10}$ is just equal to 10 times $\log n$. $(\log n)^2 = \log^2 n$ is slightly faster than a logarithm.
Finally $\log 2^n$ is simply n; but both $\log(n!)$ and $n \log n$ grow slightly faster than n; they are "Big-O" of each other:

$$n \log n \text{ is in } O(\log n!);$$

$$\log n! \text{ is in } O(n \log n).$$

It does not matter, in which order we list them.
To verify all these claims, you need to prove various limits:

$$\lim_{n\to\infty} \frac{\log\log n}{\log n} = 0,$$

$$\lim_{n\to\infty} \frac{\log n}{(\log n)^2} = 0,$$

and so on. Most of these limits are easy to find (L'Hospital's Rule and so on). With $\log n!$ you might need to use integrals to estimate the sum of $\log 1 + \log 2 + \ldots + \log n$.
Note. Unless noted otherwise, all logarithms in our course are base 2.

Question 11. Answer: `A`. Clearly the N and T should not depend on n; so they are the first quantifiers.
(B) describes a sequence of digits where some digit repeats itself infinitely often (which is true for any sequence of digits).
(C) describes the set of all sequences; one can always pick N that is larger than n, then the condition is trivially true.
(D) describes a sequence where each digit appears infinitely often.