Quiz 9: Advanced Counting

Question 1. You have a rectangular chocolate bar of $2 \times n$ squares. Denote by $a(n)$ the ways how you can divide the chocolate bar into little dominoes (rectangles 1×2). For example, $a(1) = 1$ and $a(2) = 2$. Figure 1 shows a possible tiling for the rectangle $2 \times n$, where $n = 8$.

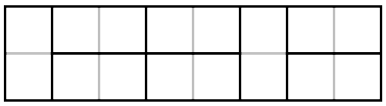

Figure 1. Domino Tiling.

In your answer write the recurrent expression of `a(n)` (express it through the previous members of the sequence: `a(n-1),...`).

Note. Assume that all the little chocolate squares are distinguishable; the cuts that differ by some symmetry (rotation or flip of the chocolate bar) are considered to be different.

Question 2. Somebody wants to find out, in how many ways it is possible to pay \$15 in a vending machine, using \$1 coins, \$1 bills, \$2 bills and \$5 bills, where the order, how you insert them into the machine matters.
In your answer write an integer number.
Note. In order to solve this, you probably want to define a recurrent sequence for b_n and find b_{15}. People sometimes use characteristic functions for such problems as well: see https://bit.ly/2Qu4Re4. Then they can solve some variations of this problem (what happens, if you have limited number of \$5 bills, etc.), but they need to use infinite power series and other calculus techniques.

Question 3. Consider a recurrent sequence:

$$\begin{cases} a_0 = 0, \\ a_1 = 1, \\ a_n = 2a_{n-1} + 2a_{n-2}, \; n \geq 2. \end{cases}$$

Find both roots of the characteristic equation r_1, r_2.
In you answer write two real numbers, round them to the nearest thousandth.

Question 4 There is a sequence $a(n)$ such that $a(0) = 0$, $a(1) = 0$, $a(2) = 1$, but its characteristic equation is $(r-1)^2(r-2) = 0$. (See https://bit.ly/3a72Ps0 where the characteristic equations with repeated roots are explained.) Find the value $a(8)$.
In your answer write a number.

Question 5 Write the following expression:
`((A / (B - (C + D))) * E) - (F + (G + H))`
in prefix (Polish) notation and also in the postfix (reverse-Polish) notation.
In your answer separate both expressions with a comma.

Question 6 Find the number of ways to parenthesize the following expression:
`A / B - C + D * E - F + G + H`
You should not assume any associativity for the operations. For example `(F + G) + H` and `F + (G + H)` are two different ways to insert parentheses. On the other hand `(F + G) + H` and `((F + (G)) + H)` is the same way, since the order of execution in both is the same.
In your answer write an integer number.

Question 7. Suppose $f(n) = 3f(n/3) + 2n$, $f(1) = 1$. Find $f(3^8)$.
In your answer write an integer number.

Answers

Question 1. Answer: `a(n)=a(n-1)+a(n-2)`
Notice that the leftmost two squares in the $2 \times n$ rectangle can be filled in two different ways:
Alternative 1: They can be filled by a single vertical domino. In this case the remaining rectangle $2 \times (n-1)$ can be filled in $a(n-1)$ ways.
Alternative 2: They can be filled by two horizontal dominoes. In this case the remaining rectangle $2 \times (n-2)$ can be filled in $a(n-2)$ ways.
The total number of ways is obtained by adding $a(n-1)$ and $a(n-2)$.
Note. The sequence $a(n)$ is a shifted Fibonacci sequence. We can easily verify that $a(n) = F_{n+1}$ for $n \geq 0$. (We define $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ is the regular Fibonacci sequence.)

Question 2. Answer: `527383`
Let us define the sequence b_n (in how many ways you can insert \$1 coins, \$1 bills, \$2 bills and \$5 bills into a vending machine so that it adds up to n dollars).
But before that we consider a simpler sequence a_n (in how many ways you can insert \$1 coins, \$1 bills, \$2 bills into a vending machine to pay n dollars – i.e. you do not use \$5 bills at all).
Sequence a_n was already discussed in the *Sample Quiz 9*, Problem 2. It is defined as a recurrent sequence:

$$\begin{cases} a_1 = 2, \\ a_2 = 5, \\ a_n = 2a_{n-1} + a_{n-2}, \; n \geq 3. \end{cases}$$

If you wish, you can also define $a_0 = 1$ (there is exactly one way how to pay \$0 – use zero instances of every coin and bill). Compute the initial members of this sequence:

$$(a_1, a_2, a_3, a_4) = (2, 5, 12, 29).$$

Notice that the sequence b_n (where we also allow \$5 bills) would have exactly same members up to b_4 (because sums up to 4 dollars cannot use any \$5 bills anyway). Further members of b_n can be defined recursively:

$$\begin{cases} b_0 = 1, \\ b_1 = 2, \\ b_2 = 5, \\ b_3 = 12, \\ b_4 = 29, \\ b_n = 2b_{n-1} + b_{n-2} + b_{n-5}, \; n \geq 5. \end{cases}$$

The last (recurrent) line means that b_n ($n \geq 5$) is a total of four different kinds of sequences:

- Some sequences start from a \$1 coin; the rest is paid in b_{n-1} different ways.
- Some sequences start from a \$1 bill; the rest is paid in b_{n-1} different ways.
- Some sequences start from a \$2 bill; the rest is paid in b_{n-2} different ways.
- Some sequences start from a \$5 bill; the rest is paid in b_{n-5} different ways.

Here are the first members $b_0, \ldots, b_{15}$ of this sequence: 1, 2, 5, 12, 29, 71, 173, 422, 1029, 2509, 6118, 14918, 36376, 88699, 216283, 527383.

Question 3. Answer:
`2.732,-0.732` or `-0.732,2.732`
The characteristic equation is

$$r^2 - 2r - 2 = 0.$$

And the roots of this square equation are $r_{1,2} = 1 \pm \sqrt{3}$. By rounding them to the nearest thousandth, we get the answer.
Note. By the way, we can also find the formula for a_n. (See *Sample Quiz 9*, Problem 3, **(B)**.)

Question 4 Answer: `247`
Rewrite the characteristic equation:

$$\begin{aligned}(r-1)^2(r-2) &= (r^2 - 2r + 1)(r-2) = \\ &= r^3 - 2r^2 + r - 2r^2 + 4r - 2 = \\ &= r^3 - 4r^2 + 5r - 2 = 0.\end{aligned}$$

We can restore the recurrent relationship from here:

$$a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}.$$

n	0	1	2	3	4	5	6	7	8
a_n	0	0	1	4	11	26	57	120	247

Question 5 Answer:
`-*/A-B+CDE+F+GH, ABCD+-/E*FGH++-`
In order to transform infix notation into postfix notation we can do this step by step. We start with the last/outermost operation (minus in our case). And leave both subexpressions in the infix form. Then we find the last/outermost operation the subexpressions and so on.

```
((A/(B-(C+D)))*E) - (F+(G+H))
- ((A/(B-(C+D)))*E) (F+(G+H))
- * (A/(B-(C+D))) E (F+(G+H))
- * / A (B-(C+D)) E (F+(G+H))
- * / A - B (C+D) E (F+(G+H))
- * / A - B + C D E (F+(G+H))
- * / A - B + C D E + F (G+H)
- * / A - B + C D E + F + G H
```

In these expressions the regular (infix) arithmetic operations are shown in black, but prefix arithmetic operations are shown in red. Postfix transformation is very similar.

Question 6 Answer: `429`
The expression `A/B-C+D*E-F+G+H` contains 7 operations and 8 operands/letters. It can be parenthesized in $C_7 = 429$ ways, where C_n is the sequence of *Catalan numbers* defined recursively:

$$\begin{cases} C_0 = 1, \\ C_{n+1} = \sum_{i=0}^{n} C_i \cdot C_{n-i} \end{cases}$$

Here are the first few members:
$C_0 = 1,\ C_1 = 1,\ C_2 = 2,\ C_3 = 5,\ C_4 = 14,\ C_5 = 42,$
$C_6 = 132,\ C_7 = 429,\ C_8 = 1430,\ C_9 = 4862,$
$C_{10} = 16796.$

Question 7 Answer: `111537`
We can compute the subsequent values, using the recursive formula.

n	$f(n)$
3^0	1
3^1	$9 = 3 \cdot 3$
3^2	$45 = 5 \cdot 9$
3^3	$189 = 7 \cdot 27$
3^4	$729 = 9 \cdot 81$
3^5	$2673 = 11 \cdot 243$
3^6	$9477 = 13 \cdot 729$
3^7	$32805 = 15 \cdot 2187$
3^8	$111537 = 17 \cdot 6561$

It is possible to prove by induction that
$f(n) = (2n + 1) \cdot 3^n$.
We can also apply this theorem:

Master Theorem: Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d$$

whenever $n = b^k$, where k is a positive integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) \text{ is } \begin{cases} O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^d \log n\right) & \text{if } a = b^d \\ O\left(n^{\log_b a}\right) & \text{if } a > b^d \end{cases}$$

In our case $a = 3$, $b = 3$, $c = 2$ and $d = 1$. Observe that $a = b^d$, therefore we get that $f(n)$ is in $O(n \log n)$.

For example, if $n = 3^8 = 6561$, then $f(n) = 6561 \cdot 17$. We clearly see both factors: 6561 grows as $O(n)$, but $17 = 2 \cdot (\log_3 6561) + 1$ grows as $O(\log n)$ (we do not care about the base of a logarithm in the Big-O notation). So their product $f(n)$ grows as $O(n \log n)$.