Worksheet 1: Boolean Logic

Question 1. Fill in the missing entries in the truth table of this proposition:

$$E = \neg(r \to \neg q) \lor (p \land \neg r).$$

p	q	r	E
T	T	T	T
T	T	F	...
T	F	T	F
T	F	F	...
F	T	T	T
F	T	F	...
F	F	T	F
F	F	F	...

Question 2. Find the Boolean expression that has this truth table:

p	q	?
T	T	F
T	F	T
F	T	T
F	F	F

(Select 1 answer):

(A) $\neg(\neg(p \land \neg p) \land \neg(\neg q \land q))$,
(B) $\neg(\neg(p \land q) \land \neg(\neg p \land \neg q))$,
(C) $\neg(\neg(p \land \neg q) \land \neg(\neg p \land q))$,
(D) $\neg((\neg p \land \neg q) \land \neg(p \land q))$.

Question 3. Determine whether the following proposition is *satisfiable*: $(\neg p \lor \neg q) \land (p \to q)$. If it is satisfiable, what are the truth values for p and q that makes it `true`.
Reminder. A Boolean expression is called *satisfiable*, if you can assign its variables to make it evaluate to `true`. In other words, it is not always `false`.

Question 4. Consider the following proposition: "Not eating vegetables is sufficient for not getting ice cream." Express it as a Boolean expression, if there are two atomic propositions:
A: "Person x eats vegetables."
B: "Person x gets ice cream."

Question 5. Determine whether the following two propositions are logically equivalent: $E_1 = p \to (\neg q \land r)$ and $E_2 = \neg p \lor \neg(r \to q)$.
If they are not equivalent, find some values p, q, r that makes E_1 different from E_2.

Question 6. Translate the given statement into propositional logic using the propositions provided:

> "On certain highways in the Washington, DC metro area you are allowed to travel on high occupancy lanes during rush hour only if there are at least three passengers in the vehicle."

Express your answer in terms of 3 atomic propositions
r: "You are traveling during rush hour."
t: "You are riding in a car with at least three passengers." and
h: "You can travel on a high occupancy lane."

Question 7a. (*Note:* In this problem "knights" always tell the truth and "knaves" always lie.)
On the island of knights and knaves you encounter two people, A and B (each of them knows everything about himself and the other person).
Person A says "B is a knave."
Person B says "At least one of us is a knight."
Determine whether each person is a knight or a knave.

Question 7b. An island has three kinds of people: knights who always tell the truth, knaves who always lie, and normals who can either tell the truth or lie. You encounter three people, A, B, and C. You know one of the three people is a knight, one is a knave, and one is a normal. Each of the three people knows the type of person each of the other two is.
A says "I am not a knight,"
B says "I am not a normal," and C says "I am not a knave."
Write a possible solution (for example, (A, B, C) = (`knight`, `knave`, `normal`) or some other permutation), or state that there are no solutions.

Answers

Question 1: Answer:

p	q	r	?
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

Note that $r = \texttt{false}$ in all the unknown slots. We can simplify the Boolean expression:

$$E \equiv \neg(r \to \neg q) \vee (p \wedge \neg r) \equiv$$
$$\equiv \neg(\texttt{false} \to \neg q) \vee (p \wedge \texttt{true}) \equiv$$
$$\equiv \neg(\texttt{true}) \vee p \equiv$$
$$\equiv \texttt{false} \vee p \equiv p.$$

Therefore we simply copy the value of p in all four places.

Question 2: Answer: **(C)**.
The truth table is identical to $p \oplus q$ (exclusive OR). To see that answer **(C)** is correct, apply De Morgan's law multiple times:

$$\neg(\neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q)) \equiv$$
$$\neg\neg(p \wedge \neg q) \vee \neg\neg(\neg p \wedge q) \equiv$$
$$(p \wedge \neg q) \vee (\neg p \wedge q).$$

The last expression is exactly the exclusive OR (either p is `true` and q is `false` or vice versa).

Question 3: Answer: Yes, $(\neg p \vee \neg q) \wedge (p \to q)$ is satisfiable.
Subexpression $(\neg p \vee \neg q)$ tells that either p or q (or both) should be `false`. So, you should take $p = \texttt{false}$ to make $p \to q$ `true` as well.

Question 4. Answer: $\neg A \to \neg B$ (or $B \to A$).
Condition C is called *sufficient* for the result R, if R is `true` whenever C is `true`. This means that **not** eating vegetables logically implies **not** getting ice cream. It is exactly $\neg A \to \neg B$.
You can rewrite it as a contrapositive, if you like: $B \to A$: getting ice cream implies having eaten vegetables. Both answers are equivalent.

Question 5. Answer: Yes, E_1 and E_2 are logically equivalent (they mean the same thing). Consequently, there is no way to assign p, q, r to make their truth values different.

- $p \to (\neg q \wedge r)$ means that either condition (p) is `false` or the conclusion $(\neg q \wedge r)$ is `true`. So, E_1 can be rewritten as $\neg p \vee (\neg q \wedge r)$.

- In E_2 the subexpression $\neg(r \to q)$ can be rewritten as $(r \wedge \neg q)$ (for the implication $r \to q$ to be `false`, you should have both $r = \texttt{true}$ and $q = \texttt{false}$, so it means $(r \wedge \neg q)$. And that is exactly the same expression we got from E_1, since $(r \wedge \neg q) = (\neg q \wedge r)$

Question 6. Answer: $(r \wedge h) \to t$
"Only if" means **necessity**. Namely, $(r \wedge h)$ implies t, or $(r \wedge h) \to t$ (Ability to use a high occupancy road during a rush hour implies at least three people in the car.)
You can rewrite this as a contrapositive (and expand $(r \wedge h)$ with De Morgan's law). Say, "If you do not have three people in your car, then it is either not a high occupancy road, or it is not a rush hour." This is an equivalent statement $\neg t \to (\neg r \vee \neg h)$.

Question 7a. Answer: A is a knave; B is a knight.
We sort cases.
(1) Assume that A is a knight. He says that B is a knave (and that must be true, since A does not lie). But then B should lie, when he says "At least one of us is a knight". The opposite of "At least one" (≥ 1) is "less than one" (< 1); so in this case there should be no knights at all. This is a contradiction. Therefore A cannot be a knight.
(2) Assume that A is a knave. Since he says that B is a knave, B should be a knight. And moreover, since B says that "at least one of us is a knight", then it is still consistent, because he is a knight himself.

Question 7b. TBD (I can explain this in the class, if you wish.)