Worksheet 4: Sets

Question 1. We define three sets in the universe U of integer numbers between 1 and 70 (inclusive):

$$\begin{cases} K_2 &= \{x \in U \mid 2 \mid x\}, \\ K_5 &= \{x \in U \mid 5 \mid x\}, \\ K_7 &= \{x \in U \mid 7 \mid x\}, \end{cases}$$

Find the size of the following sets. Here $|X|$ denotes the number of elements in a finite set (BTW, $|X|$ is also the notation for the cardinality of an infinite set).
Note. It was not intentional, but vertical bar: | in this exercise happens to be used in three different ways: It is a separator when defining sets K_2; it is used to denote divisibility; it also denotes set cardinality.

$\lvert K_2 \cup K_5 \rvert$	...
$\lvert K_2 \cap K_7 \rvert$	...
$\lvert \overline{K_7} \rvert$	...
$\lvert \overline{K_2 \cap K_5} \rvert$	...
$\lvert K_2 \cap K_5 \cap K_7 \rvert$	...
$\lvert K_2 \cap K_5 \cap \overline{K_7} \rvert$	...
$\lvert K_2 \cup K_5 \cup K_7 \rvert$	...
$\lvert \overline{K_2 \cup K_5 \cup K_7} \rvert$	...

Question 2. Find a counterexample to refute the following predicate expression:
$(\exists x \in U,\ P(x)) \land (\exists x \in U,\ Q(x)) \rightarrow$
$\rightarrow \exists x \in U,\ (P(x) \land Q(x))$.
Here $P(x)$ is true iff P is a full square (a square of some integer number), $Q(x)$ is true iff x is divisible by 5, and U is the set of all integers from the interval $[120; 130]$.
Note. The three x's in this formula refer to three unrelated (local) variables. If it looks confusing, you can rewrite it like this:
$(\exists x_1 \in U,\ P(x_1)) \land (\exists x_2 \in U,\ Q(x_2)) \rightarrow$
$\rightarrow \exists x_3 \in U,\ (P(x_3) \land Q(x_3))$.
(A) Identify the variables which you need to pick for your counter-example.
(B) Pick the values for these variables to make the above statement false.

Question 3. Determine the cardinality of the following sets (some finite number? equal to $|\mathbb{N}|$? equal to $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$? equal to $|\mathcal{P}(\mathbb{R})|$?)

(A) The set of positive real numbers from $(0; 1)$ with decimal representation containing only digits 0 and 1?

(B) The set of positive real numbers from $(0; 1)$ with decimal representation containing only digits 0 and 1 (and it is known that the number of 0s is finite)?

(C) The set of positive real numbers from $(0; 1)$ that are fully periodic decimal fractions with a period of 2020 digits?

(D) The set of positive real numbers from $(0; 1)$ that have decimal representation without any digits "9"?

(E) The set of all irrational $x \in (0; 1)$ such that x^3 is rational?

(F) Ordered pairs of real numbers x_1, x_2 such that $x_1, x_2 \in (0; 1)$.

(G) Finite sequences of real numbers: $x_1, \ldots, x_n$, and all $x_i \in (0; 1)$? (Here n can be any positive integer)?

(H) Infinite sequences of real numbers from $(0; 1)$: $\{x_n\}$: $x_1, x_2, x_3, \ldots$.

Question 4. Let $f(x) = \left\lfloor \dfrac{x^3}{3} \right\rfloor$. Find $f(S)$ if S is:

(A) $S = \{2, 1, 0, 1, 2, 3\}$.

(B) $S = \{0, 1, 2, 3, 4, 5\}$.

(C) $S = \{1, 5, 7, 11\}$.

Is function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ injective? Is it surjective? If it is not, mention counterexamples to show this.

Question 5. Determine, if the given set is a powerset of some other set. If yes, which one?

(A) $\{\emptyset,\ \{\emptyset\},\ \{a\},\ \{\{a\}\},\ \{\{\{a\}\}\},\ \{\emptyset, a\},\ \{\emptyset, \{a\}\},\ \{\emptyset, \{\{a\}\}\},$
$\{a, \{a\}\},\ \{a, \{\{a\}\}\},\ \{\{a\}, \{\{a\}\}\},\ \{\emptyset, a, \{a\}\},\ \{\emptyset, a, \{\{a\}\}\},$
$\{\emptyset, \{a\}, \{\{a\}\}\},\ \{a, \{a\}, \{\{a\}\}\},\ \{\emptyset, a, \{a\}, \{\{a\}\}\}\}$.

(B) $\{\emptyset, \{a\}\}$.

(C) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$.

(D) $\{\emptyset, \{a\}, \{\emptyset\}, \{a, \emptyset\}\}$.

(E) $\{\emptyset, \{a, \emptyset\}\}$.

Question 6. Given two sets $A = \{x, y\}$ and $B = \{x, \{x\}\}$, check, if statements are true or false:

(A) $x \subseteq B$.

(B) $\emptyset \in \mathcal{P}(B)$.

(C) $\{x\} \subseteq A - B$.

(D) $|\mathcal{P}(A)| = 4$.

Question 7. We define functions $g : A \rightarrow A$ and $f : A \rightarrow A$, where $A\{1, 2, 3, 4\}$ by listing all argument-value pairs:

$$g = \{(1, 4), (2, 1), (3, 1), (4, 2)\}, \quad f = \{(1, 3), (2, 2), (3, 4), (4, 1)\}.$$

Find these functions by listing their argument/value pairs (or establish that they do not exist).

(A) Find $f \circ g$.

(B) Find $g \circ f$.

(C) Find $g \circ g$.

(D) Find $g \circ (g \circ g)$.

(E) Find f^{-1}.

(F) Find g^{-1}.

Question 8. Find these sums:

(A) $1/4 + 1/8 + 1/16 + 1/32 + \ldots$.

(B) $2 + 4 + 8 + 16 + 32 + \ldots + 2^{28}$.

(C) $2 - 4 + 8 - 16 + 32 - \ldots - 2^{28}$.

(D) $1 - 1/2 + 1/4 - 1/8 + 1/16 - \ldots$.

Question 9. Find an appropriate $O(g(n))$ for each function $f(n)$ defined below (pick your $g(n)$ to be the slowest growing among the functions such that $f(n)$ is in $O(g(n))$).

(A) $f(n) = 1^2 + 2^2 + \ldots + n^2$.

(B) $f(n) = \dfrac{3n - 8 - 4n^3}{2n - 1}$.

(C) $f(n) = \sum_{k=1}^{n} k^3$.

(D) $f(n) = \dfrac{6n + 4n^5 - 4}{7n^2 - 3}$.

(E) $f(n) = \sum_{k=2}^{n} k \cdot (k - 1)$.

(F) $f(n) = 3n^2 + 8n + 7$

Question 10. For the given functions, find an optimal $O(g(n))$; find C and n_0 (from the definition $|f(n)| < C \cdot |g(n)|$ as long as $n > n_0$).

(A) $f(n) = 3n^4 + \log_2 n^8$.

(B) $f(n) = \sum_{k=1}^{n} (k^3 + k)$.

(C) $f(n) = (n + 2) \log_2(n^2 + 1) + \log_2(n^3 + 1)$.

(D) $f(n) = n^3 + \sin n^7$.

Question 11. This is a Python fragment; variable `n` can become very large; `t` is some fixed parameter. Denote by $f(n)$ the number of operations depending on the variable `n`, where an operation is an addition or a multiplication, or raising to the power 2. Find the slowest growing $g(n)$ so that $f(n)$ is in $O(g(n))$.

```
sum = 0
for i in range(1,n+1):
    for j in range(1,n+1):
        sum += (i*t + j*t + 1)**2
```

Question 12. There are two functions $f, g : \mathbb{R} \to \mathbb{R}$ defined for all real numbers and taking real values. Find, which predicate logic expressions describe a statement that is logically equivalent to the English sentence "The function $f(n)$ is in $O(g(n))$".
Note. There may be multiple correct answers.

(A) $\forall n \in \mathbb{R}\ \exists n_0 \in \mathbb{R}\ \exists C \in \mathbb{R}$,
$(n > n_0 \to |f(n)| \leq C \cdot |g(n)|)$.

(B) $\exists n_0 \in \mathbb{R}\ \forall n \in \mathbb{R}\ \exists C \in \mathbb{R}$,
$(n > n_0 \to |f(n)| \leq C \cdot |g(n)|)$.

(C) $\exists n_0 \in \mathbb{R}\ \exists C \in \mathbb{R}\ \forall n \in \mathbb{R}$,
$(n > n_0 \to |f(n)| \leq C \cdot |g(n)|)$.

(D) $\exists n_0 \in \mathbb{R}\ \exists C \in \mathbb{R}\ \forall n \in \mathbb{R}$,
$(n > n_0 \to f(n) \leq C \cdot |g(n)|)$.

(E) $\exists n_0 \in \mathbb{R}\ \exists C \in \mathbb{R}\ \forall n \in \mathbb{R}$,
$(n > n_0 \to |f(n)| \leq C \cdot g(n))$.

(F) $\exists n_0 \in \mathbb{R}\ \exists C \in \mathbb{R}\ \forall n \in \mathbb{R}$,
$(n \geq n_0 \to |f(n)| < C \cdot |g(n)|)$.

(G) $\exists n_0 \in \mathbb{Z}^+\ \exists C \in \mathbb{Z}^+\ \forall n \in \mathbb{R}$,
$(n > n_0 \to |f(n)| \leq C \cdot |g(n)|)$.

Answers

Question 1. Answer:

$\lvert K_2 \cup K_5 \rvert$	42
$\lvert K_2 \cap K_7 \rvert$	5
$\lvert \overline{K_7} \rvert$	60
$\lvert \overline{K_2 \cap K_5} \rvert$	63
$\lvert K_2 \cap K_5 \cap K_7 \rvert$	1
$\lvert K_2 \cap K_5 \cap \overline{K_7} \rvert$	6
$\lvert K_2 \cup K_5 \cup K_7 \rvert$	46
$\lvert \overline{K_2 \cup K_5 \cup K_7} \rvert$	24

To find $|K_2 \cup K_5 \cup K_7|$ we might use inclusion-exclusion principle:

$$|K_2 \cup K_5 \cup K_7| = (35 + 14 + 10) - (7 + 5 + 2) + 1 = 46.$$

Question 2. Answer: TBD

Question 3. Answer: TBD

Question 4. Answer: TBD

Question 5. Answer: TBD

Question 6. Answer: TBD

Question 7. Answer: TBD

Question 8. Answer: TBD

Question 9. Answer: TBD

Question 10. Answer: TBD

Question 11. Answer: TBD

Question 12. Answer: TBD