Worksheet 9: Advanced Counting

Question 1 (Rosen7e, Ch.8, Q4). Denote by a_n the number of ways to go down an n-step staircase if you go down 1, 2, or 3 steps at a time. (In other words, a_n denotes the number of ways how n can be expressed as a sum of $1, 2, 3$, if the order in the sum matters.)
Define a_n as a recurrent sequence. Assume that the sequence starts from a_1.

Question 2 (Rosen7e, Ch.8, Q11). A vending machine accepts only \$1 coins, \$1 bills, and \$2 bills. Let a_n denote the number of ways of depositing n dollars in the vending machine, where the order in which the coins and bills are deposited matters.
(A) Find a recurrence relation for a_n and give the necessary initial condition(s).
(B) Find an explicit formula for a_n by solving the recurrence relation in part **(A)**.

Question 3 (Rosen7e, Ch.8, Q16-Q20). For each item solve the recurrence relation (characteristic equation or simply guess the pattern for the terms):
(A) $a_n = a_{n-2}$, $a_0 = 2$, $a_1 = -1$.
(B) $a_n = 2a_{n-1} + 2a_{n-2}$, $a_0 = 0$, $a_1 = 1$.
(C) $a_n = 3na_{n-1}$, $a_0 = 2$.
(D) $a_n = a_{n-1} + 3n$, $a_0 = 5$.
(E) $a_n = 2a_{n-1} + 5$, $a_0 = 3$.

Question 4 (Rosen7e, Ch.8, Q24) Assume that the characteristic equation for a homogeneous linear recurrence relation with constant coefficients is $(r + 2)(r + 4)^2 = 0$.
(A) Describe the form for the general solution to the recurrence relation.
(B) Define a recurrent sequence that leads to this characteristic equation.

Question 5 (Rosen7e, Ch.8, Q27) The Catalan numbers C_n count the number of strings of n pluses (+) and n minuses (−) with the following property: as each string is read from left to right, the number of pluses encountered is always at least as large as the number of minuses.
(A) Verify this by listing these strings of lengths 2, 4, and 6 and showing that there are C_1, C_2, and C_3 of these, respectively.
(B) Explain how counting these strings is the same as counting the number of ways to correctly parenthesize strings of variables.
Note. Catalan numbers

$$C_0 = 1,\ C_1 = 1,\ C_2 = 2,\ C_3 = 5,\ C_4 = 14, \ldots$$

are defined as $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$.

Question 6 (Rosen7e, Ch.8, Q28, Q29)
(A) What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + F(n)$ have when $F(n) = 2^n$?
(B) What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + F(n)$ have when $F(n) = n2^n$?

Question 7 (Rosen7e, Ch.8, Q32). Consider the recurrence relation $a_n = 2a_{n-1} + 3n$.
(A) Write the associated homogeneous recurrence relation.
(B) Find the general solution to the associated homogeneous recurrence relation.
(C) Find a particular solution to the given recurrence relation.
(D) Write the general solution to the given recurrence relation.
(E) Find the particular solution to the given recurrence relation when $a_0 = 1$.

Question 8 (Rosen7e, Ch.8, Q37). Suppose $f(n) = f(n/3) + 2n$, $f(1) = 1$. Find $f(27)$.

Question 9 (Rosen7e, Ch.8, Q53-Q63) Find the coefficient of x^8 in the power series of each of the function:
(A) $(1 + x^2 + x^4)^3$.
(B) $(1 + x^2 + x^4 + x^6)^3$.
(C) $(1 + x^2 + x^4 + x^6 + x^8)^3$.
(D) $(1 + x^2 + x^4 + x^6 + x^8 + x^{10})^3$.
(E) $(1 + x^3)^{12}$.
(F) $(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)$.
(G) $1/(1 - 2x)$.
(H) $x^3/(1 - 3x)$.
(I) $1/(1 - x)^2$.
(J) $x^2/(1 + 2x)^2$.
(K) $1/(1 - 3x^2)$.

Answers

Question 1. For staircases with $n = 1, 2, 3$ steps we can enumerate all possible sums and count them.
1-step staircases: 1.
2-step staircases: 1 + 1, 2.
3-step staircases: 1 + 1 + 1, 1 + 2, 2 + 1, 3.

$$\begin{cases} a_1 = 1, \\ a_2 = 2, \\ a_3 = 4, \\ a_n = a_{n-1} + a_{n-2} + a_{n-3}, \ n \geq 4. \end{cases}$$

Why *an* can be expressed as sum of a_{n-1}, a_{n-2}, a_{n-3}?
Let us consider the first movement downstairs. It can be either 1 step down (then the remaining steps can be made in a_{n-1} different ways), or 2 steps down (then the remaining steps can be made in a_{n-2} different ways), or 3 steps down (then the remaining steps can be made in a_{n-3} different ways).
All these options do not intersect (since any moving downstairs starts with either 1, 2 or 3), therefore we simply add all these ways together: $a_{n-1} + a_{n-2} + a_{n-3}$.

Question 2.
(A) Let us find the initial conditions (values a_1 and a_2) and the recurrence relation (how to express a_n via the the two previous values).

$$\begin{cases} a_1 = 2, \\ a_2 = 5, \\ a_n = 2a_{n-1} + a_{n-2}, \ n \geq 3. \end{cases}$$

The initial conditions can be verified by noticing that 1 dollar can be paid in two ways (a coin or a bill); 2 dollars can be paid in five ways (either one \$2 bill or any of the following: coin+coin, coin+bill, bill+coin, bill+bill).
If $n \geq 3$, then a_n either starts by a \$1 coin (plus paying the remaining sum in a_{n-1} different ways) or by a \$1 bill (plus paying the remaining sum in a_{n-1} different ways), or by a \$2 bill (plus paying the remaining sum in a_{n-2} different ways). By adding together these mutually exclusive options we get

$$a_n = a_{n-1} + a_{n-1} + a_{n-2} = 2a_{n-1} + a_{n-2}.$$

Question 3.
(A) Characteristic equation: $r^2 - 1 = 0$. The roots are $r_1 = -1$ and $r_2 = 1$. So, the general form of this sequence is

$$a_n = A \cdot (-1)^n + B \cdot 1^n.$$

To get $a_0 = 2$ and $a_1 = -1$, we need $A = \frac{3}{2}$ and $B = \frac{1}{2}$. Therefore,

$$a_n = \frac{3}{2} \cdot (-1)^n + \frac{1}{2}.$$

This expression would generate our alternating sequence

$$2, -1, 2, -1, 2, -1, \ldots$$

(B) The characteristic equation is

$$r^2 - 2r - 2 = 0.$$

And the roots of this square equation are $r_{1,2} = 1 \pm \sqrt{3}$. We need to find constants A and B such that the formula

$$a_n = A \cdot r_1^n + B \cdot r_2^n$$

is correct and we get $a_0 = 0$ and $a_1 = 1$. By substituting $n = 0$ and $n = 1$ we get this system:

$$\begin{cases} A + B = 0 \\ A(1 + \sqrt{3}) + B(1 - \sqrt{3}) = 1 \end{cases}$$

We get that $B = -A$ and $2A\sqrt{3} = 1$. Therefore $A = \frac{1}{2\sqrt{3}}$ and $B = -\frac{1}{2\sqrt{3}}$. The formula to compute a_n is therefore this:

$$a_n = \frac{1}{2\sqrt{3}} \left(1 + \sqrt{3}\right)^n - \frac{1}{2\sqrt{3}} \left(1 - \sqrt{3}\right)^n.$$

By plugging into this formula numbers from $[0; 10]$, we can compute the first members of the sequence:

$$0, 1, 2, 6, 16, 44, 120, 328, 896, 2448, 6688, \ldots$$

(C) a_n can be obtained from $a_0 = 2$, if we multiply n times by 3 (i.e. by 3^n), and also by $1 \cdot 2 \cdot \ldots \cdot n = n!$. Therefore, we get

$$a_n = 2 \cdot 3^n \cdot n!.$$

(D) a_n can be obtained from $a_0 = 5$ by adding some members of an arithmetic progression:

$$a_n = a_0 + (3 + 6 + \ldots + (3n)) = 5 + \frac{3n(n+1)}{2}.$$

Question 4
(A) $a_n = A \cdot (-2)^n + B \cdot (-4)^n + C \cdot n \cdot (-4)^4$.
(B) $(r+2)(r+4)^2 = (r+2)(r^2 + 8r + 16) = r^3 + 8r^2 + 16r + 2r^2 + 16r + 32 = r^3 + 10r^2 + 32r + 32$. From that cubic equation we can get this recurrent relationship:

$$a_n = -10a_{n-1} - 32a_{n-2} - 32a_{n-3}.$$

Initial conditions (values for the first three members, say, a_1, a_2, a_3 can be selected in any way).