Worksheet 10: Binary Relations

Question 1 (A reminiscence about variance)
Assume that you want to encode six-letter alphabet $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}, \mathtt{f}\}$ and transmit it over a computer network. You assign 2 or 3 bit codes to these letters:

a	00
b	01
c	100
d	101
e	110
f	111

For example, the 11-bit sequence `"10100100110"` means `"dace"`. Denote by X the random variable – the number of bits used to encode a single letter. (All 6 letters have equal probabilities.)
Find $E(X)$ and $V(X)$.
Write them as two fractions: `P1/Q1,P2/Q2`
(Separate the fractions by comma, do not leave any spaces.)

Question 2 (Rosen7e, Ch.9, Q10-Q23).
Determine whether the binary relation is: (1) reflexive, (2) symmetric, (3) antisymmetric, (4) transitive. Express your answer as as a 4-letter string of T/F (true/false values that are answer to these 4 questions). For example, `TFTT` etc.

(A) The relation R on $\{1, 2, 3, \ldots\}$ where aRb means $a \mid b$.

(B) The relation R on $\{w, x, y, z\}$ where $R = \{(w, w), (w, x), (x, w), (x, x), (x, z), (y, y), (z, y), (z, z)\}$.

(C) The relation R on $\mathbb{Z}$ where aRb means $|a - b| \leq 1$.

(D) The relation R on $\mathbb{Z}$ where aRb means $a \neq b$.

(E) The relation R on $\mathbb{Z}$ where aRb means that the units digit of a is equal to the units digit of b.

(F) The relation R on the set of all subsets of $\{1, 2, 3, 4\}$ where SRT means $S \subseteq T$.

(G) The relation R on the set of all people where aRb means that a is younger than b.

(H) The relation R on the set $\{(a, b) \mid a, b \in \mathbb{Z}\}$ where $(a, b)R(c, d)$ means $a = c$ or $b = d$.

Question 3 (Rosen7e, Ch.9, Q35-Q38).
Construct a matrix of the relations defined below. Output the matrix as a list of lists:
`[[a11,a12,...],[a21,a22,...],...]`

(A) R on $\{1, 2, 3, 4, 6, 12\}$ where aRb means $a \mid b$.

(B) R on $\{1, 2, 3, 4, 6, 12\}$ where aRb means $a \leq b$.

(C) R^2, where R is the relation on $\{1, 2, 3, 4\}$ such that aRb means $|a - b| \leq 1$.

Question 4 (Rosen7e, Ch.9, Q42).
Define

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

determine if R is: (1) reflexive (2) symmetric (3) antisymmetric (4) transitive. Express your answer as as a 4-letter string of T/F (true/false values that are answer to these 4 questions). For example, `TFTT` etc.

Question 5 (Rosen7e, Ch.9, Q47).
Let A be the set of all positive divisors of 60 (including 1 and 60 itself). Draw the Hasse diagram for the relation R on A where aRb means $a \mid b$.

Question 6 (Rosen7e, Ch.9, Q51).
Find the transitive closure of R if

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Question 7 (Rosen7e, Ch.9, Q59).
Find the join of the 3-ary relation:

```
{ (Wages,MS410,N507),
  (Rosen,CS540,N525),
  (Michaels,CS518,N504),
  (Michaels,MS410,N510) }
```

and the 4-ary relation:

```
{ (MS410,N507,Monday,6:00),
  (MS410,N507,Wednesday,6:00),
  (CS540,N525,Monday,7:30),
  (CS518,N504,Tuesday,6:00),
  (CS518,N504,Thursday,6:00) }
```

with respect to the last two fields of the first relation and the first two fields of the second relation.

Question 8 (Rosen7e, Ch.9, Q69-Q71). Give an example of a relation or state that there are none.
(A) A relation on $\{a, b, c\}$ that is reflexive and transitive, but not antisymmetric.
(B) A relation on $\{1, 2\}$ that is symmetric and transitive, but not reflexive.
(C) A relation on $\{1, 2, 3\}$ that is reflexive and transitive, but not symmetric.

Question 9 (Rosen7e, Ch.9, Q73).
Suppose $|A| = 7$. Find the number of reflexive, symmetric binary relations on A.

Answers

Question 1. The random variable X takes value $x_1 = 2$ (with probability $p_1 = \frac{1}{3}$) and value $x_2 = 3$ (with probability $p_2 = \frac{2}{3}$). We can compute:

$$E(X) = x_1 p_1 + x_2 p_2 = \frac{8}{3}.$$

$$V(X) = (x_1 - E(X))^2 p_1 + (x_2 - E(X))^2 p_2 = \frac{2}{9}.$$

Question 2.

(A) `TFTT`

(B) `TFFF`

(C) `TTFF`

(D) `FTFF`

(E) `TTFT`

(F) `TFTT`

(G) `FFTT`

(H) `TTFF`

Question 3.
(A) Divisibility on the set $\{1, 2, 3, 4, 6, 12\}$

$$M_R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(B) Relation $\leq$ on the set $\{1, 2, 3, 4, 6, 12\}$

$$M_R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(C) Relation R^2, where aRb iff $|a - b| \leq 1$.

$$M_R = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

The only pairs that do not belong to R^2 are $(1;4)$ and $(4;1)$.

Question 4. Answer: `FFFF`
Imagine that the relation R is defined on a set of these 4 elements: a, b, c, d.

- R is not reflexive, since bRb is false (the matrix has $m_{22} = 0$).
- R is not symmetric, since bRa does not imply aRb (the matrix has $m_{12} = 0$, but $m_{21} = 1$).
- R is not antisymmetric, since aRc and cRa both hold, but $a \neq c$.
- R is not transitive, since aRc and cRb, but aRb is not true.

Question 5:
Hasse diagram connects only those numbers a, b where a divides b, and there is no third number in-between (such that $a \mid c$ and $c \mid b$). For example, 1 and 2 are connected, but 1 and 4 are not (because the relation $1 \mid 4$ can be inferred from $1 \mid 2$ and $2 \mid 4$).

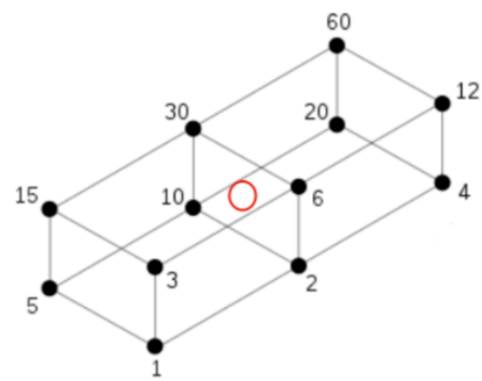

Figure 1. Hasse diagram: Divisors of 60.

Note that the Hasse diagram for divisibility is centrally symmetric. (This is true for any set of divisors for some number.)

Question 6:

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Denote the elements of the set by a, b, c, d. We want to find, what are the "relational paths" between them:

- Path aRc, cRb, bRd adds (a, b), (a, d) to the transitive closure.
- Path bRa, aRc, cRb adds (b, c), (b, b) to the transitive closure.
- Path cRb, bRd adds (c, d) to the transitive closure.
- Path cRb, bRa adds (c, a) to the transitive closure.
- Path dRb, bRa, aRc adds (d, a), (d, c).
- Path dRb, bRd adds (d, d).

Here is the matrix of the transitive closure after all the new pairs are added:

$$M_{R*} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Question 7
TBD (see Section 9.2.4 of the textbook (Definition 4, page 615)). This problem is based entirely on applying the definition.

Question 8
(A) TBD
(B). Yes. We can have a relation R which is never satisfied. It is symmetric and also transitive (since xRy and yRz can never happen, so we do not need to care about xRz).
(C). TBD.

Question 9 Answer: `2097152`
The matrix M of any relation R on a set of 7 elements has 49 entries. The entries on the main diagonal ($m_{11}, m_{22}, \ldots, m_{77}$) should all equal 1 ($R$ is reflexive). Also, any entry m_{ij} above the main diagonal ($i < j$ - row number is less than the column number) is symmetric to some entry below the diagonal m_{ji} (where $i > j$).

Therefore we can freely choose only those m_{ij} that are above the main diagonal; everything else is predetermined. There are $1 + 2 + \ldots + 6 = 21$ such elements in the matrix. The total number of ways to choose them is $2^{21} = 2097152$.