Worksheet 11: Graphs

Question 1.
(A) K_n (the complete graph on n vertices) has ...... edges and ...... vertices.
(B) $K_{m,n}$ (the complete bipartite graph on sets of sizes m, n) has ...... edges and ...... vertices.
(C) W_n (wheel graph on n vertices – n-gonal pyramid viewed from above) has ...... edges and ...... vertices.
(D) Q_n (n-dimensional cube) has ...... edges and ...... vertices.

Question 2 (Rosen7e, Ch.10, Q15-Q17).
(A) The length of the longest simple circuit in K_5 is ....
(B) The length of the longest simple circuit in W_{10} is ....
(C) The length of the longest simple circuit in $K_{4,10}$ is ....
Note. A simple circuit in (Rosen2019) is defined as a circular sequence of vertices $v_0, v_1, \ldots, v_n = v_0$, where each two neighboring vertices are connected by an edge (and it does not contain any edge more than once). It can return to the same vertex multiple times.

Question 3 (Rosen7e, Ch.10, Q19-Q24). In each example find the dimensions of a matrix; and number of 0s and 1s in it: Find X, Y, Z, T.
(A) The adjacency matrix for $K_{m,n}$ has size (rows times columns) $X \times Y$; it has Z 0's and T 1's.
(B) The adjacency matrix for K_n has size $X \times Y$; it has Z 0's and T 1's.
(C) The adjacency matrix for C_n has size $X \times Y$; it has Z 0's and T 1's.
(D) The adjacency matrix for Q_4 has size $X \times Y$; it has Z 0's and T 1's.
(E) The incidence matrix for W_n has size $X \times Y$; it has Z 0's and T 1's.
(F) The incidence matrix for Q_5 has size $X \times Y$; it has Z 0's and T 1's.
Note. Adjacency matrix is a square matrix of size $|V| \times |V|$, but incidence matrix is a rectangular matrix of size $|V| \times |E|$.

Question 4 (Rosen7e, Ch.10, Q28-Q31).
(A) List all positive integers n such that K_n has an Euler circuit; what is its length in terms of n?
(B) List all positive integers n such that Q_n has an Euler circuit.

Question 5 (Rosen7e, Ch.10, Q43).
If G is a planar connected graph with 12 regions and 20 edges, then G has ...... vertices.

Question 6 (Rosen7e, Ch.10, Q44).
If G is a planar connected graph with 20 vertices, each of degree 3, then G has ...... regions.

Question 7 (Rosen7e, Ch.10, Q45).
If a regular graph G has 10 vertices and 45 edges, then each vertex of G has degree ....
Note. A *regular graph* is a graph where all vertices have the same degree.

Question 8 (Rosen7e, Ch.10, Q59-Q82).
(A) A simple graph with 6 vertices, whose degrees are $2, 2, 2, 3, 4, 4$.
(B) A simple graph with 8 vertices, whose degrees are $0, 1, 2, 3, 4, 5, 6, 7$.
(C) A simple graph with degrees $1, 2, 2, 3$.
(D) A simple graph with degrees $2, 3, 4, 4, 4$.
(E) A simple graph with degrees $1, 1, 2, 4$.
(F) A simple digraph with indegrees $0, 1, 2$ and outdegrees $0, 1, 2$.
(G) A simple digraph with indegrees $1, 1, 1$ and outdegrees $1, 1, 1$.
(H) A simple digraph with indegrees $0, 1, 2, 2$ and outdegrees $0, 1, 1, 3$.
(I) A simple digraph with indegrees $0, 1, 2, 4, 5$ and outdegrees $0, 3, 3, 3, 3$.
(J) A simple digraph with indegrees $0, 1, 1, 2$ and outdegrees $0, 1, 1, 1$.
(K) A simple digraph with indegrees: $0, 1, 2, 2, 3, 4$ and outdegrees: $1, 1, 2, 2, 3, 4$.
(L) A simple graph with 6 vertices and 16 edges.
(M) A connected simple planar graph with 5 regions and 8 vertices, each of degree 3.
(N) A graph with 4 vertices that is not planar.
(O) A planar graph with 10 vertices.
(P) A planar graph with 8 vertices, 12 edges, and 6 regions.
(Q) A planar graph with 7 vertices, 9 edges, and 5 regions.

Question 9 (Rosen7e, Ch.10, Q108).
Use Dijkstras Algorithm to find the shortest path length between the vertices a and z in these weighted graphs.
(A)

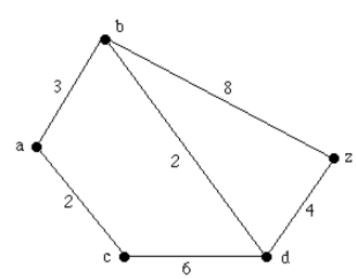

Figure 1. Weighted graph 1.

(B)

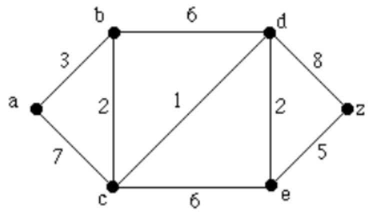

Figure 2. Weighted graph 2.

Question 10 (Rosen7e, Ch.10, Q113).
The picture at the right shows the floor plan of an office. Show that it is impossible to plan a walk that passes through each doorway exactly once, starting and ending at A.

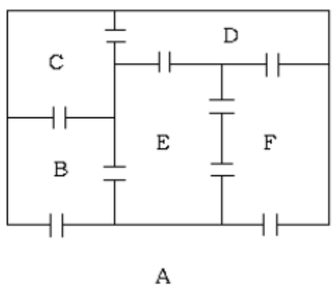

Figure 3. Floor plan.

Hall's Marriage Theorem (Rosen2019, p.772)
The bipartite graph $G = (V, E)$ with partition of vertices into 2 disjoint sets $V = X \cup Y$ has a *maximum matching* that saturates X iff for all $A \subseteq X$ we have $|X| \leq |N(X)|$.
Note. A *matching* in a graph is a set of of edges such that no two edges share a common endpoint. A *maximum matching* is matching containing the greatest number of edges. And a matching *saturates* a set X, if each vertex $v \in X$ belongs to some matching edge.

Question 11 (Königs Marriage Theorem)
Prove that if all the vertices of a bipartite graph have the same degree, then it has a perfect matching.
(Quines2017, p.11); https://cjquines.com/files/halls.pdf
Note. A matching is *perfect*, if it saturates all vertices (every vertex has a pair).

Question 12
We have a regular deck of 52 playing cards, with exactly 4 cards of each of the 13 ranks. The cards have been randomly dealt into 13 piles, each with 4 cards in it. Prove that there is a way to take a card from each pile so that after we take a card from every pile, we have exactly a card of every rank.
(Quines2017, p.11).

Answers

Question 1. Answer:

(A) K_n has $\frac{n(n-1)}{2}$ edges and n vertices.
(B) $K_{m,n}$ has $m \cdot n$ edges and $m + n$ vertices.
(C) W_n has $2n$ edges and $n + 1$ vertices.
(D) Q_n has $n \cdot 2^{n-1}$ edges and 2^n vertices.
The number of edges in Q_n could be first expressed by a recurrent formula (and then proven by mathematical induction).

Question 2. Answer: TBD

Question 3. Answer: TBD

Question 4. Answer: TBD

Question 5. Answer: TBD

Question 6. Answer: TBD

Question 7. Answer: TBD

Question 8. Answer: TBD

Question 9. Answer: TBD

Question 10. Answer: TBD

Question 11. Answer: TBD

Question 12. Answer: TBD