# Final Review: Relations

Discrete Structures
(Final scheduled for Wednesday, April 28, 2021)

**You must justify all your answers to recieve full credit**

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These sample problems help to prepare for the final. The first two topics on *relations* also need some concepts related to sets, functions, images and preimages of functions and relations.

## 1 Binary Relations

Properties, functional relations, equivalence and order.

1.(a). Given a desription of a binary relation as a bipartite graph, matrix, list, a set-builder expression (or just verbal explanation), find its description in another form.

1.(b). Given a binary relation determine if it is a function, an injective, surjective or bijective function.

1.(c). Given a description of a binary relation between countable sets, find images of particular values in its domain (including values of functional relations).

1.(d). Given some relations find which pairs belong to them.

1.(e). Given a binary relation determine if it is reflexive, symmetric, antisymmetric or transitive.

1.(f). Given a relation, check if it is an equivalence relation, get the underlying set's partition into equivalence classes (sometimes find one representative in every equivalence class).

1.(g). Given a relation, check if it is a partial or total order relation.

1.(a). Given a description of some binary relation, convert the relation into some other representation:

(a) The binary relation "divides" is defined on set $A = \{1, 2, 3, 4, 5, 6\}$. Represent it in the set/roster notation (i.e. list all pairs). Represent it into the set-builder notation. Represent it as a digraph.

(b) The binary relation "less than or equal to" is defined on two sets $A = \{1, 3, 5\}$ and $B = \{2, 4\}$. Represent it as a bipartite graph. Represent it as a matrix.

(c) The binary relation $R = \{(a, b) \mid |a - b| \leq 2\}$ is defined on the set $A = [0, 5] \subseteq \mathbf{R}$. Show its Plane/Cartesian graph.

(d) The binary relation $R = \{(a, b) \mid a - b \equiv 3 \pmod 8\}$ is defined on the set $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Represent the set $A$ as vertices of a regular octagon and show this relation as a digraph.

1.(b). For the given relations determine if it is a functional relation (and if it is functional, is it also injective, surjective and/or bijective).

(a) Let us have sets $A = \mathbf{N}$ and $B = \{0, 1, 2\} \times \mathbf{N}$. A relation $R \subseteq A \times B$ is defined as follows:

$$R = \{(a, (b_1, b_2)) \in A \times B \mid a = 3b_2 + b_1\}.$$

Here $\mathbf{N} = \{0, 1, 2, \ldots\}$ is the set of all natural numbers.

(b) Let us have sets $A = \mathbf{N}$ and $B = \{0, 1\} \times \{0, 1\}$. A relation $R \subseteq A \times B$ is defined as follows:

$$R = \{(a, (b_1, b_2)) \in A \times B \mid a \equiv b_1 \pmod 2 \ \wedge \ (a - b_1) = 2b_2 \pmod 4\}.$$

1.(c). Let $A = \{\epsilon, a, b, aa, ab, ba, bb, aaa, \ldots\}$ be the set of all strings written using just two letters $a$ and $b$ (and $\epsilon$ is the empty string). We define a binary relation $R$ on $A \times \mathbf{N}$ (where $\mathbf{N}$ is the set of all natural numbers $\mathbf{N} = \{0, 1, 2, \ldots\}$) that contains all pairs $(s, n)$, where $n$ is the number of string $s$ in the shortlex ordering:

$$R = \{(\epsilon, 0), (a, 1), (b, 2), (aa, 3), (ab, 4), (ba, 5), (bb, 6), (aaa, 7), \ldots.\}$$

Find the inverse image $R^{-1}(\{10, 20, 30\})$ of the set $\{10, 20, 30\} \subseteq \mathbf{N}$ for the relation $R$.

1.(d). Consider the following binary relations:

$$\begin{aligned}
R_1 &= \{(a,b) \in \mathbf{Z} \times \mathbf{Z} \mid a \leq b\}, \\
R_2 &= \{(a,b) \in \mathbf{Z} \times \mathbf{Z} \mid a > b\}, \\
R_3 &= \{(a,b) \in \mathbf{Z} \times \mathbf{Z} \mid a = b \vee a = b\}, \\
R_4 &= \{(a,b) \in \mathbf{Z} \times \mathbf{Z} \mid a = b\}, \\
R_5 &= \{(a,b) \in \mathbf{Z} \times \mathbf{Z} \mid a = b + 1\}, \\
R_6 &= \{(a,b) \in \mathbf{Z} \times \mathbf{Z} \mid a + b \leq 3\}.
\end{aligned}$$

(a) Find a pair that is in $R_1$, but not in $R_3$ (or vice versa).
(b) Find a pair that is in $R_1$, but not in $R_6$ (or vice versa).
(c) Find a pair that is in $R_2$, but not in $R_5$ (or vice versa).
(d) Find a pair that is in $R_3$, but not in $R_4$ (or vice versa).
(e) Find a pair that is in $R_5$, but not in $R_6$ (or vice versa).

1.(e). Let $A = \{0, 1, 2, 3, 4\}$ and let $R$ be a binary relation on this set:

$$R = \{(a, b) \mid a^3 \equiv b \pmod 5\}.$$

(a) Is $R$ a reflexive relation?
(b) Is $R$ an antisymmetric relation?
(c) Is $R$ a functional relation? (And is it also injective, surjective and/or bijective?)

1.(f). Let $A = \{1, 2, 3, 4, 5, 6\}$ and $R \subseteq A \times A$ given by relation matrix:

$$M_R = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

(a) Write the matrix of another relation $R^*$ that is the reflexive closure of $R$.
(b) Show that $R^*$ is an equivalence relation.
(c) Compute $A/R^*$ (the partition of $A$ induced by the relation $R^*$).
(d) Draw $R^*$ as a directed graph.

1.(g). Let $A = \{1, 2, 3, 4, 5, 6\}$ and let $R \subseteq A \times A$ be a binary relation with these pairs:

$$R = \{(1,1), (1,3), (2,2), (2,5), (3,3), (4,4), (5,4), (5,5), (6,6)\}.$$

Let $R^*$ be the transitive closure of $R$. Is $R^*$ a partial order relation?

## 2 Manipulating Relations

Closures, $n$-ary relations and relational algebra.

2.(a). Given some relations or functions, find their composition (represent with bipartite graphs or as matrix multiplications).

2.(b). Given a binary relation on a single set, compute its powers.

2.(c). (Warshall Algorithm) Given a binary relation, compute its transitive closure; show the steps.

2.(d). Given a relation, find its reflexive, symmetric, transitive closure (also multiple closures).

2.(e). Given $n$-ary relations, apply the 6 relational algebra operations; show relation tables or just determine their size.

2.(f). Given $n$-ary relations, compute inner join, left outer join, right outer join and full outer join of relations.

2.(a). Let $A = \{a, b\}$, $B = \{0, 1, 2\}$ and $C = \{x, y\}$ be three sets and let us define two relations $R \subseteq A \times B$ and $S \subseteq B \times C$ (in set/roster notation):

$$R = \{(a, 0), (a, 2), (b, 1)\},$$
$$S = \{(0, x), (0, y), (1, y), (2, y)\}.$$

(a) Show $R$ and $S$ in matrix notation (or bipartite graph notation, if you like it better).

(b) Show the composition $S \circ R$ in matrix notation (or bipartite graph notation).

2.(b). Let $A = \{1, 2, 3\}$ and $R$ is a binary relation on $A$ defined by this matrix:

$$M_R = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute the power $R^3$ of this relation.

2.(c). Given the relation on set $A = \{0, 1, 2, 3\}$ shown in Figure 1, use Warshall algorithm to compute its transitive closure.

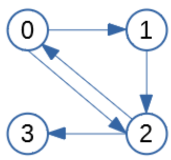


Figure 1: Relation for the Warshall algorithm.

2.(d). Let $A = \{a, b, c\}$ be a set and let $R$ be a binary relation on $A$ defined by the following matrix:

$$M_R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Find the matrix of $R_1$ which is the transitive closure of $R$.

(b) Find the matrix of $R_2$ which is the reflexive closure of $R_1$.

(c) Find the matrix of $R_3$ which is the symmetric closure of $R_2$.

(d) Is the relation $R_3$ (obtained after all three successive closures) itself reflexive, symmetric and transitive?

2.(e). Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be a set and $R$ is a ternary relation:

$$R = \{(a, b, c) \mid \gcd(a, b) = c\}.$$

(a) Find the number of rows in this relation. I.e. how many triples $(a, b, c)$ satisfy this definition

| $a$ | $b$ | $c$ |
|---|---|---|
| ... | ... | ... |
| ... | ... | ... |

(b) Let $a, b, c$ be the column names in this ternary relation $R$. We compute two more relations from $R$:

$$R_1 = \sigma_{(c=3 \vee c=5)}(R),$$
$$R_2 = \pi_{a,b}(R_1).$$

Represent both $R_1$ and $R_2$ with their relational tables.

2.(f). We have two relational tables $R_1$ and $R_2$.

$R_1 =$

| $a$ | $b$ | $id$ |
|---|---|---|
| $A1$ | $B1$ | 11 |
| $A2$ | $B2$ | 11 |
| $A3$ | $B3$ | 13 |

$R_2 =$

| $id$ | $c$ | $d$ |
|---|---|---|
| 11 | $C1$ | $D1$ |
| 12 | $C2$ | $D2$ |
| 13 | $C3$ | $D3$ |

(a) Compute the inner join $R_1 \bowtie R_2$ on column $id$.

(b) Compute the right join of the same relations on column $id$.