Final Review: Probability

Discrete Structures
(Final scheduled for Wednesday, April 28, 2021)

You must justify all your answers to recieve full credit

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6 Probability

Keywords: Single events, multiple events, complements, independence, conditional probability, Bayes' theorem, Bernoulli trials.

6.1 Topics

6.(a) Define the sample space of an experiment, describe the events and compute their probabilities using Laplace's definition.

6.(b) Compute probabilities of derived events (complementary, intersection, union, etc.).

6.(c) Analyze the probabilities of the outcomes of a probabilistic 2-player game (such as Monty Hall).

6.(d) Identify where Bayes' theorem should be applied and apply it.

6.2 Sample questions

6.(a) A class of 700 medical residents is graduating. The residents compete for 300 positions, and all are equally likely to get one of the positions.

i. How large is the sample space for the outcome of the competition?
ii. What is the probability that a particular resident gets any position?
iii. What is the probability that three particular residents each get a position?
iv. What is the probability that at least one of four particular residents does not get a position?

6.(b) Five cards are drawn from a standard deck of cards at random.

i. What is the probability that at least one is a spade?
ii. What is the probability that all five have the same suit?
iii. What is the probability that there are exactly two face cards and exactly two black cards?

6.(c) Two players play the following game with one coin:

- Player A selects a string of 3 letters, using only `H` (heads) or `T` (tails).
- Player B selects a different string of 3 letters, using the same letters `H` or `T`.
- They toss the coin and record the outcomes as a string of letters `H` and `T`. They stop tossing the coin when **both** of their selected strings appeared. The player who selected the string that occured before the other player's string wins.

Suppose that Player A selected `"HHT"` and Player B selected `"THH"`. Let X be the random variable that has value $n \in \mathbf{N}_{\geqslant 3}$ if the last three of n coin tosses make the Player A's string `"HHT"` (but this `"HHT"` did not appear earlier).

(a) Compute $P(X = n)$ for $n = 3, 4, 5, 6$.

(b) Compute the expected value $E(X)$.

(c) Find the probability that Player A wins.

6.(d) 1000 people are tested for a disease with a test that gives the correct answer 90% of the time. It is known that 40% of the people have the disease. You take the test.

i. What is the probability that the test says you have the disease?

ii. Suppose that the test says you have the disease. What is the probability that you actually do have the disease?

7 Random Variables

Keywords: Expected value, variance, distributions, independence, Chebyshev's inequality.

7.1 Topics

7.(a) Identify the Bernoulli, binomial, and geometric distributions.

7.(b) Given a problem description, define random variables for its probabilistic model, define its distribution as a function.

7.(c) Given a distribution for a discrete random variable X, compute $E(X)$ and $V(X)$.

7.(d) Given a random variable X, estimate probabilities of X being in some interval by Chebyshev's inequality.

7.2 Sample questions

7.(a).
i. You pick a card at random from a deck. What is the probability you pick an ace?

ii. You roll a die 10 times. What is the probability you get exactly 8 sixes?

iii. You choose numbers at random between 1 and 100 inclusive until you pick a number larger than 55. What is the expected value of the number of times you have to choose?

7.(b). Somebody tosses a fair coin 9 times and records the result as a string of letters `T` (tails) and `H` (heads). The result is a palindrome, that is, the same when read left-to-right or right-to-left. Let X be the random variable whose value is the number of heads `H` minus the number of tails `T`.

(a) Find the possible values of X.

(b) For every value n of X find its probability $P(X = n)$.

(c) Can you come up with a continuous function $f(n)$ so that $f(n) = P(X = n)$ for all n?

7.(c). Let X be a discrete random variable with the following distribution f:

x	-1	2	3	7
$f(x)$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{2}{5}$

Compute the expected value and the variance of X.

7.(d). Let X be a random variable on the sample space $\{1, 2, \ldots, 9\}$ with probability distribution function $P(X = x) = \begin{cases} x/25 & x \leqslant 5 \\ (10-x)/25 & x > 5 \end{cases}$. Compute a lower bound for $P(3 \leqslant X \leqslant 7)$.