Final

Discrete Structures
April 28, 2021

You must justify all your answers to recieve full credit

---

1. (10 points) Let $A = \{0,1,2,3,4,5,6,7,8,9\}$. Define a binary relation R on A by

$$R = \left\{(a,b) \in A \times A \ \mid \ a^2 \equiv b^2 \pmod{10}\right\}.$$

(a) Represent this binary relation as a directed graph.

Full squares can give the following 6 remainders when divided by 10: $0,1,4,5,6,9$. 0^2 ends with 0, 1^2 and 9^2 end with digit 1 and so on. Whenever two digits have squares that give the same remainder when divided by 10, we connect them with an arrow:

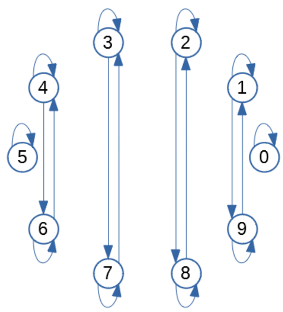


Figure 1: Relation as a directed graph.

$\square$

(b) Is this relation an equivalence relation on A?

Because of the congruence properties (modulo 10) every a^2 is congruent to itself (reflexivity); if $a^2 \equiv b^2$ then also $b^2 \equiv a^2$ (symmetry), also $a^2 \equiv b^2$ and $b^2 \equiv c^2$ implies $a^2 \equiv b^2$ (transitivity).
Therefore it is also an equivalence relation. (Equivalence classes, except $\{0\}$ and $\{5\}$ contain two digits each.) $\square$

(c) Is this relation a partial order on A?

As we established before, the relation R is both reflexive and transitive.
But it is not anti-symmetric, since $1^2 \equiv 9^2 \pmod{10}$ and also $9^2 \equiv 1^2 \pmod{10}$. Meanwhile, elements 1 and 9 are two different elements in the set A; we have $1 \neq 9$.

$\square$

(d) Is this relation a functional relation on A?

We need to check, if the relation for each $a \in A$ allows to find a unique $b \in A$ such that $(a,b) \in R$. (This would mean a well-defined function.)

This is false, because most elements $a \in A$ (except 0 and 5) are in relation with two elements (e.g. 1 is in relation with itself and also with 9). Therefore, if we try to convert R into a function, is not clear, whether $f(1) = 1$ or $f(1) = 9$. $\square$

2. (10 points) Let $X = \{a, b, c, d, e, f, g\}$. Define a binary relation S on X by

$$S = \{(a, d), (b, e), (c, f), (d, g), (e, a), (f, b), (g, c)\}.$$

(a) Express the binary relation S as a matrix.

In this matrix all the 7 rows (and also all the 7 columns) are represented by the elements of the set X (its elements are in the original order). $\{a, b, c, d, e, f, g\}$. In this matrix, say $m_{14} = 1$, because $a \in X$ is in relation with $d \in X$ ($(a, d) \in S$). We get this matrix:

$$M_S = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\square$

(b) Express the composition binary relation $S^2 = S \circ S$ as a matrix.

The composition of relation S with itself (in other words, the power of the relation S^2) contains a pair (x, z) iff there exists some y such that $(x, y) \in S$ and $(y, z) \in S$. Since every element only connects to a single element, we jump two arrows (every arrow jumps 3 steps ahead). Therefore, in S^2 we end up jumping 6 steps ahead (or one step back, if we regard $\{a, b, c, d, e, f, g\}$ as circular). See Figure 2.

$$M_{S^2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

$\square$

(c) Find the transitive closure of the binary relation S. How many pairs does the transitive closure of S contain?

By definition, transitive closure S^t of a relation S includes all the pairs (a_0, a_n) such that there exists a "path" of n binary relations such that $(a_0, a_1) \in S$, $(a_1, a_2) \in S$, $\ldots$, $(a_{n-1}, a_n) \in S$.

Since 7 (the size of the set) and the "step length" 3 are mutually prime, transitive closure includes every pair $(x, y) \in S$ such that $x, y \in X$. For example, $(a, b) \in S^t$,

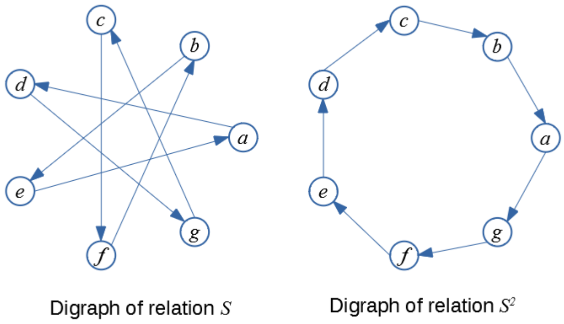


Figure 2: Relation S and S^2 (composition with itself).

since the pairs (a, d), (d, g), (g, c), (c, f) and (f, b) all belong to S. (To make a step of length 1 one can take 5 steps of length 3 on the circle with seven elements.) By repeating this you can step ahead by any other step size (including step size 0, if you take seven steps of length 3).

We get a digraph with 7 vertices and all possible edges (including self-loops). The total number of edges is $7 \cdot 7 = 49$. $\square$

3. (10 points) Let n be a positive integer such that its units digit[1] is 3, $n+1$ is divisible by 3, and $n+2$ is divisible by 7.

   (a) Write a system of congruences corresponding to the three conditions given on n.

   Units digit shows the remainder, if n is divided by 10.
   If $n+1$ is divisible by 3, then n has remainder 2 when divided by 3.
   If $n+2$ is divisible by 7, then n has remainder 5 when divided by 7.
   Let us write the system of congruences:

$$\begin{cases} n \equiv 3 \pmod{10} \\ n \equiv 2 \pmod{3} \\ n \equiv 5 \pmod{7} \end{cases}$$

   As 10, 3 and 7 are mutually relative primes, there exists a solution for the system (Chinese Remainder theorem). $\square$

   (b) Find a number n that satisfies these three conditions.

   We can solve two congruences first, and then add one more. Let us find the solution to a smaller system:

$$\begin{cases} n \equiv 3 \pmod{10} \\ n \equiv 2 \pmod{3} \end{cases}$$

   We build the sequence of all numbers congruent to 3 modulo 10 (having units digit 3):

$$3, 13, \underline{23}, 33, 43, \underline{53}, 63, 73, \underline{83}, 93 \ldots$$

   After every three numbers in this sequence there is one congruent to 2 modulo 3 (underlined numbers 23, 53, 83, and so on). We got that these are numbers congruent to

[1]The “units digit” is the number farthest to the right when n is written in decimal notation.

23 modulo $30 = 10 \cdot 3$. Now replace the first two congruences with the new congruence (modulo 30) and add the remaining congruence (modulo 7):

$$\begin{cases} n \equiv 23 \pmod{30} \\ n \equiv 5 \pmod{7} \end{cases}$$

Once again list the arithmetic progression that matches the first congruence:

$$23, 53, 83, 113, 143, 173, 203, 233, \ldots$$

Now list the remainders of this sequence modulo 7 (until we find remainder 5):

$$23 \equiv 2,\ 53 \equiv 4,\ 83 \equiv 6,\ 113 \equiv 1,\ 143 \equiv 3,\ 173 \equiv 5,\ 203 \equiv 0,\ 233 \equiv 2,\ \ldots \pmod{7}$$

We see that 173 has the right remainder ($\equiv 5$) modulo 7. □

(c) Find the general solution for the system of congruences from part (a).

Since $10 \cdot 3 \cdot 7 = 210$, we can generalize the solution obtained in part (b) as follows:

$$n \equiv 173 \pmod{210}.$$

Indeed, the remainders (modulo 10, modulo 3, and modulo 7) do not change if the numbber 173 is incremented by $210k$ for any integer $k \in \mathbf{Z}$. □

4. (10 points) You may use the extended Euclidean algorithm or Blankinship's algorithm for this question.

(a) Find the inverse of the integer 6 modulo 19.

Note that 19 is a prime number; hence 6 and 19 are relative primes; and inverse of 6 must exist.
We run the Blankinship's algorithm for numbers 6 and 19 to find an integer solution to $6x + 19y = 1$ (Bezout identity):

$$\left(\begin{array}{c|cc} 6 & 1 & 0 \\ 19 & 0 & 1 \end{array}\right) \xrightarrow{L2:=L2-3L1} \left(\begin{array}{c|cc} 6 & 1 & 0 \\ 1 & -3 & 1 \end{array}\right) \xrightarrow{L1:=L1-6L2} \left(\begin{array}{c|cc} 0 & 19 & -6 \\ 1 & -3 & 1 \end{array}\right)$$

Look at the 2nd row. We see that 1 can be expressed as $(-3) \cdot 6 + 1 \cdot 19 = 1$. Therefore

$$(-3) \cdot 6 \equiv 1 \pmod{19}.$$

We can also add 19 to the number (-3) on the left side:

$$(-3 + 19) \cdot 6 \ \equiv\ 16 \cdot 6 \ \equiv\ 1 \pmod{19}.$$

Thus 16 is the inverse of 6 (modulo 19). □

(b) Use the inverse from (a) to solve for x in the congruence equation $6x \equiv 7 \pmod{19}$.

Multiply both sides of the congruence with 16 (the inverse of 6):

$$16 \cdot 6x \equiv 16 \cdot 7 \pmod{19}.$$

$$1 \cdot x \equiv 112 \pmod{19}.$$
$$x \equiv 17 \pmod{19}.$$

$x = 17$ is the solution to the congruence equation. □

5. (10 points) A parking lot with ordered spaces is being built for cars and bicycles. There are two types of bicycles that take up a single parking space each, and three types of cars that take up two parking spaces each. Let p_n be the number of different ways to take up $n \geqslant 1$ parking spaces.

   (a) Find the values p_1 and p_2.

   It is immediate that $p_1 = 2$ and $p_2 = 7$. □

   (b) Find a recurrence relation for p_n.

   To compute p_n, we note that we can either fill the last space in one of two ways with a bicycle (that is, in $2p_{n-1}$ ways), or fill the last two spaces in one of three ways with a car (that is, in $3p_{n-2}$ ways). Hence the recurrence relation is

$$p_n = 2p_{n-1} + 3p_{n-2}.$$

□

6. (15 points) Consider the recurrence relation $a_n = 5a_{n-2} + 4a_{n-1}$, with initial conditions $a_0 = 1$ and $a_1 = 2$.

   (a) Give the first 5 terms of the sequence $\{a_n\}_{n=1}^{\infty}$.

   The first five terms are

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 2 \\ a_2 &= 5 \cdot 1 + 4 \cdot 2 = 13 \\ a_3 &= 5 \cdot 2 + 4 \cdot 13 = 62 \\ a_4 &= 5 \cdot 13 + 4 \cdot 62 = 65 + 248 = 313 \end{aligned}$$

□

   (b) What are the characteristic equation and the characteristic roots of this relation?

   The characteristic equation is

$$0 = r^2 - 4r - 5 = (r - 5)(r + 1),$$

   and so its roots are $r_1 = 5$ and $r_2 = -1$. □

   (c) Give the closed form of the solution to this recurrence equation.

   We know that the general solution is $a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n$, for

$$\begin{aligned} \alpha_1 &= \frac{a_1 - a_0 \cdot r_2}{r_1 - r_2} = \frac{2 - 1 \cdot (-1)}{5 - (-1)} = \frac{3}{6} = \frac{1}{2}, \\ \alpha_2 &= a_0 - \alpha_1 = 1 - \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

Hence the general solution is

$$a_n = \frac{1}{2} \cdot 5^n + \frac{1}{2} \cdot (-1)^n = \frac{5^n + (-1)^n}{2}.$$

$\square$

7. (15 points) Consider the set $Z = \{1, 2, 3, \ldots, 221\}$.

   (a) How many ways are there to choose 3 elements from Z?

   There are $\binom{221}{3} = \frac{221!}{3!218!} = \frac{221 \cdot 220 \cdot 219}{3 \cdot 2} = 221 \cdot 110 \cdot 73 = 1774630$ ways. $\square$

   (b) How many ways are there to choose 3 elements from Z so that none are divisible by 3?

   We are not allowed to choose multiples of 3, and in Z there are 73 multiples of 3. Hence the number of ways is $\binom{221-73}{3} = \binom{148}{3} = \frac{148!}{3!145!} = \frac{148 \cdot 147 \cdot 146}{3 \cdot 2} = 74 \cdot 49 \cdot 146 = 529396.$ $\square$

   (c) Suppose you choose n elements from Z. How large must n be so that you are guaranteed to have chosen 3 consecutive elements?

   Consider this way to split numbers from 221 into 74 groups (three elements each; the last group has only two elements):

$$(1, 2, 3),\ (4, 5, 6),\ \ldots,\ (217, 218, 219),\ (220, 221)$$

   If you choose the first two elements from each group ($1, 2$, and $4, 5$, and $7, 8$, ..., and $220, 221$), then you have no three consecutive elements and the selection has $2 \cdot 74 = 148$ elements. The counterexample shows that 148 elements is not enough.

   As soon as you choose 149 elements, at least one of the 74 groups must be selected with all three elements (as picking just two would give only $2 \cdot 74 < 149$). But such triplet ($(1, 2, 3)$ or $(4, 5, 6)$ etc.) is a group of three consecutive numbers.

   Hence n must be at least 149 to guarantee three consecutive numbers. $\square$

8. (15 points)

   (a) What is the coefficient of x^7y^{11} in the expansion of $(3x - y)^{18}$?

   The binomial theorem tells us that

$$(3x + y)^{18} = \sum_{k=0}^{18} \binom{18}{k} (3x)^{18-k}(-y)^k,$$

   and for $k = 11$ we have $\binom{18}{11}(3x)^7(-y)^{11}$. The coefficient of x^7y^{11} is

$$\binom{18}{11} \cdot 3^7 \cdot (-1)^{11} = -\frac{18! \cdot 3^7}{11!7!} = -69599088.$$

$\square$

   (b) Consider the permutations $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{smallmatrix}\right)$ and $\rho = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{smallmatrix}\right)$. Find a permutation τ so that $\tau \circ \sigma = \rho$.

Here we go element-by-element in σ and find where each needs to go so that we get ρ at the end. That is:

$$\begin{array}{ccccc} 1 & \xrightarrow{\sigma} & 2 & \xrightarrow{\tau} & 1 \\ 2 & \xrightarrow{\sigma} & 4 & \xrightarrow{\tau} & 3 \\ 3 & \xrightarrow{\sigma} & 1 & \xrightarrow{\tau} & 2 \\ 4 & \xrightarrow{\sigma} & 3 & \xrightarrow{\tau} & 5 \\ 5 & \xrightarrow{\sigma} & 5 & \xrightarrow{\tau} & 4 \end{array}$$

Hence we get τ to be $\tau = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{smallmatrix}\right)$. $\square$

9. (15 points) Suppose there are three events, A, B, C, and at least one of them must happen. The probabilities of two of them happening are given in the table below (on the diagonal the probability of each happening is given).

$P(\cdot \cap \cdot)$	A	B	C
A	$11/20$	$8/20$	$6/20$
B	$8/20$	?	$9/20$
C	$6/20$	$9/20$	$11/20$

(a) Are events A and C independent? Why or why not?

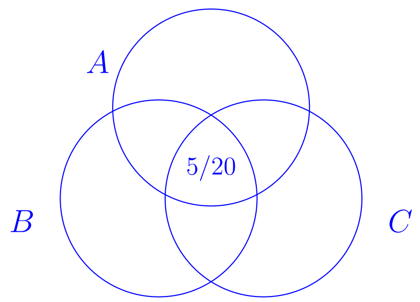
They are not independent, because $P(A) = \frac{11}{20}$ and $P(C) = \frac{11}{20}$, so then $P(A) \cdot P(C) = \frac{121}{400}$. However, $P(A \cap C) = \frac{6}{20} = \frac{120}{400} \neq \frac{121}{400}$, and so the condition for independence is not satisfied. $\square$

(b) From the table, what is the largest and smallest possible value of $P(A \cap B \cap C)$?

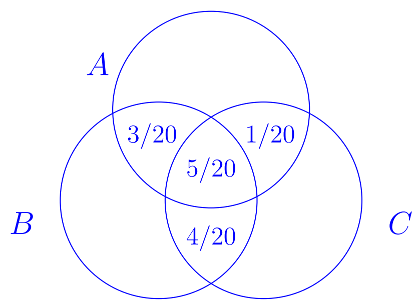
Note that probability can not be negative, and $P(A \cap B \cap C)$ is bounded above by each of $P(A \cap B)$, $P(A \cap C)$, $P(B \cap C)$. Hence $P(A \cap B \cap C) \in [0, 6/20]$. $\square$

(c) You are given that $P(A \cap B \cap C) = 5/20$. Draw the Venn diagram representing the three events A, B, C, and compute $P(B)$.

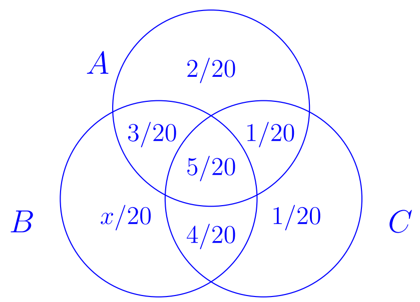
The Venn diagram with this new piece of information is given below.



Comparing with $P(A \cap B)$, $P(A \cap C)$, $P(B \cap C)$ we can fill in three more areas.



Comparing with $P(A)$ and $P(C)$ we can fill in two more areas.



We are given that at least one of the events must happen, so the sum of all the known numbers plus the remaining area must equal to 1. That is,

$$1 = \frac{x+3+5+4+2+1+1}{20} = \frac{x+16}{20} \implies 20 = x + 16,$$

so $P(B) = \frac{4+3+5+4}{20} = 16/20$. $\square$

10. (15 points) 2000 people are tested for a disease with a test that gives a yes/no answer[2], and gives it correctly answer 90% of the time. It is known that 20% of the people have the disease. You do not know if you have the disease.

(a) How many people without the disease will receive an answer saying they have the disease?

There are $(1 - \frac{2}{10}) \cdot 2000 = \frac{8}{10} \cdot 2000 = 1600$ people without the disease. If the test is accurate 90% of the time, then $(1 - \frac{9}{10}) \cdot 1600 = \frac{1}{10} \cdot 1600 = 160$ people without the disease will receive an answer saying they do have the disease. $\square$

(b) What is the probability that the test says you have the disease?

Let X be the event that you have the disease. Let Y be the event that the test gives the correct answer. The probability that the test says you have the disease is

$$\begin{aligned} P(X) \cdot P(Y) + P(\overline{X}) \cdot P(\overline{Y}) &= \frac{2}{10} \cdot \frac{9}{10} + \left(1 - \frac{2}{10}\right) \cdot \left(1 - \frac{9}{10}\right) \\ &= \frac{18}{100} + \frac{8}{10} \cdot \frac{1}{10} \\ &= \frac{18+8}{100} \\ &= \frac{26}{100} \\ &= 26\%. \end{aligned}$$

$\square$

(c) Suppose that the test says you have the disease. What is the probability that you actually do have the disease?

This is an application of Bayes' theorem. Let Z be the event that the test says you have the disease. We are asked to find

$$P(X|Z) = \frac{P(Z|X) \cdot P(X)}{P(Z)} = \frac{\frac{9}{10} \cdot \frac{2}{10}}{\frac{26}{100}} = \frac{18}{26} = \frac{9}{13}.$$

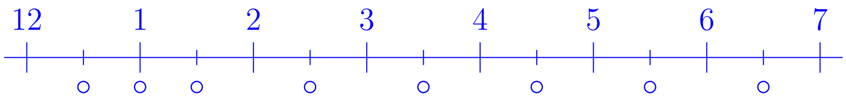
$\square$

[2] "Yes" means you have the disease, "no" means you do not have the disease.

11. (15 points) Neb has a clock that chimes every half hour, chiming as many times as the number on the clock on the hour, and chiming once on the half hour. Neb goes to sleep just after midnight and wakes up just before 7 AM. But he also wakes up twice during the night, and hears a single chime each time. Let X be the random variable representing the time of the second chime.

(a) Compute $P(X \leqslant 2{:}30 \text{ AM})$. You may assume that waking up at any moment during the night is equally likely.

Between midnight and 7 AM there are 8 occurences of a single chime:



Altogether, there are $\binom{8}{2} = \frac{8!}{2!6!} = 28$ combinations of two chimes. For one of these the second chime is at 1 AM. For two of these the second chime is at 1:30 AM. And for 3 of these the second chime is at 2:30 AM. Hence $P(X \leqslant 2{:}30\text{AM}) = \frac{1+2+3}{28} = \frac{6}{28} = \frac{3}{14}$. $\square$

(b) Give an expression for the expected value $E(X)$. You do not need to evaluate it.

The expected value will have 7 terms, representing all the single chimes from 1 AM to 6:30 AM. The probability of one of these 7 being the second chime is the number of chimes that occur before it, divided by 28. That is,

$$\begin{aligned}
E(X) &= \frac{1}{28} \cdot 1 + \frac{2}{28} \cdot 1.5 + \frac{3}{28} \cdot 2.5 + \frac{4}{28} \cdot 3.5 + \frac{5}{28} \cdot 4.5 + \frac{6}{28} \cdot 5.5 + \frac{7}{28} \cdot 6.5 \\
&= \frac{1}{28}\left(1 + \frac{2 \cdot 3}{2} + \frac{3 \cdot 5}{2} + \frac{4 \cdot 7}{2} + \frac{5 \cdot 9}{2} + \frac{6 \cdot 11}{2} + \frac{7 \cdot 13}{2}\right) \\
&= \frac{1}{28} \cdot \frac{2 + 6 + 15 + 28 + 45 + 66 + 91}{2} \\
&= \frac{1}{28} \cdot \frac{253}{2} \\
&= \frac{253}{56} \\
&\approx 4.518 \\
&= 4{:}31 \text{ AM}
\end{aligned}$$

Here we must convert 60-minute hours into decimal hours. $\square$

12. (15 points) Suppose you have two dice, one with 6 sides (having the numbers $1, \ldots, 6$) and one with 8 sides (having the numbers $1, \ldots, 8$). Both dice are fair – their numbers roll with equal probabilities. You roll both at the same time. Let X be the random variable equal to the sum of numbers rolled on both dice.

   (a) Find the variance $V(X)$.

   Let X_1 be the random variable equal to the value on the first die, and X_2 the random variable equal to the value on the second die. It is immediate that $X = X_1 + X_2$, and so we use Bienyame's formula to get $V(X) = V(X_1) + V(X_2)$. The variance is computed using the expected value, which we find to be

$$\begin{aligned} E(X_1) &= \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6 \\ &= \frac{1+2+3+4+5+6}{6} \\ &= \frac{6\cdot 7/2}{6} \\ &= \frac{7}{2} \end{aligned}$$

   and

$$\begin{aligned} E(X_2) &= \frac{1}{8}\cdot 1 + \frac{1}{8}\cdot 2 + \frac{1}{8}\cdot 3 + \frac{1}{8}\cdot 4 + \frac{1}{8}\cdot 5 + \frac{1}{8}\cdot 6 + \frac{1}{8}\cdot 7 + \frac{1}{8}\cdot 8 \\ &= \frac{1+2+3+4+5+6+7+8}{8} \\ &= \frac{8\cdot 9/2}{8} \\ &= \frac{9}{2}. \end{aligned}$$

   Then the variance is

$$V(X_1) = \left(1-\frac{7}{2}\right)^2\cdot\frac{1}{6} + \left(2-\frac{7}{2}\right)^2\cdot\frac{1}{6} + \left(3-\frac{7}{2}\right)^2\cdot\frac{1}{6} + \left(4-\frac{7}{2}\right)^2\cdot\frac{1}{6} + \left(5-\frac{7}{2}\right)^2\cdot\frac{1}{6} + \Big(6-$$

   Method 2. Use the definition of variance: $V(X) = E((X - E(X))^2)$.
   The values of $X - E(X) = X - 8$ are $-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6$. Their probabilities were found in Homework 11, Q1.

$(x_i - 8)^2$	36	25	16	9	4	1	0	1	4	9	16	25	36
$p(X = x_i)$	$\frac{1}{48}$	$\frac{2}{48}$	$\frac{3}{48}$	$\frac{4}{48}$	$\frac{5}{48}$	$\frac{6}{48}$	$\frac{6}{48}$	$\frac{6}{48}$	$\frac{5}{48}$	$\frac{4}{48}$	$\frac{3}{48}$	$\frac{2}{48}$	$\frac{1}{48}$

   Multiply all the columns and add them together to find $V(X) = E((X - E(X))^2)$. (The terms in this sum are symmetric from both ends):

$$36\cdot\frac{1}{48} + 25\cdot\frac{2}{48} + 16\cdot\frac{3}{48} + 9\cdot\frac{4}{48} + 4\cdot\frac{5}{48} + 1\cdot\frac{6}{48} + 0\cdot\frac{6}{48} + 1\cdot\frac{6}{48} + 4\cdot\frac{5}{48}\ldots + 36\cdot\frac{1}{48} =$$

$$= 2\cdot\left(36\cdot\frac{1}{48} + 25\cdot\frac{2}{48} + 16\cdot\frac{3}{48} + 9\cdot\frac{4}{48} + 4\cdot\frac{5}{48} + 1\cdot\frac{6}{48}\right) + 0\cdot\frac{6}{48} = \frac{392}{48} = 8\frac{1}{6}.$$

   Method 3. For an n-sided fair die (X_n taking values $1, 2, \ldots, n$ with probabilities $1/n$) compute variance using the formula $V(X) = E(X^2) - (E(X))^2$:

$$V(X_n) = \frac{1}{n}\sum_{k=1}^{n} k^2 - \left(\frac{1}{n}\sum_{k=1}^{n} k\right)^2 = \frac{1}{n}\cdot\frac{n(n+1)(2n+1)}{6} - \left(\frac{1}{n}\cdot\frac{n(n+1)}{2}\right)^2 =$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{2n^2+3n+1}{6} - \frac{n^2+2n+1}{4} = \frac{n^2-1}{12}.$$

Therefore, a fair 6-sided die has variance $V(X_6) = \frac{6^2-1}{12} = \frac{35}{12}$, but a fair 8-sided die has variance $V(X_8) = \frac{8^2-1}{12} = \frac{63}{12}$.

Since both dice rolls X_6 and X_8 are independent, the variance of their sum $X = X_6 + X_8$ is the sum of variances:

$$V(X) = V(X_6) + V(X_8) = \frac{35}{12} + \frac{63}{12} = \frac{98}{12} = 8\frac{1}{6}.$$

□

(b) Use Chebyshev's inequality to estimate the probability $p(X \in \{2, 3, 13, 14\})$.

Let us write the original form of Chebyshev's inequality (Rosen2019, p.517):

$$p\left(|X - E(X)| \geq r\right) \leq V(X)/r^2.$$

We have $E(X) = 8$. We have $X \in \{2, 3, 13, 14\}$ iff $|X - E(X)| = |X - 8| \geq 5$; the random variable X must differ from its expected value by 5 or more.

Plug in the values $r = 5$ and $V(X) = 8\frac{1}{6}$:

$$p\left(|X - 8| \geq 5\right) \leq \frac{V(X)}{r^2} = \frac{8\frac{1}{6}}{5^2} \approx 0.3266667.$$

□

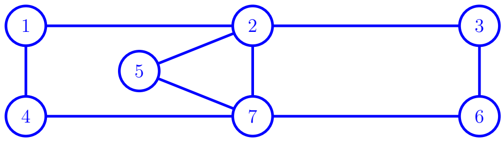
(c) Compute the actual probability $p(X \in \{2, 3, 13, 14\})$.

All pairs that add up to $2, 3, 13$ or 14: $(1;1)$, $(1;2)$, $(2;1)$, $(5;8)$, $(6;7)$, $(6;8)$. All have the same probability $\frac{1}{48}$. The total probability for six cases is $\frac{6}{48} = \frac{1}{8} = 0.125$. □

13. (15 points) Consider the following undirected graph G, represented as an adjacency matrix.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

(a) Draw the graph.



☐

(b) Is the graph bipartite? If yes, give a bipartition. If no, justify why.

No, because there are cycles $(2, 5, 7)$ and $(1, 2, 5, 7, 4)$ of odd length. If a graph has a cycle of odd length, it cannot be bipartite, as was shown in a homework exercise. ☐

(c) Does the graph have an Euler circuit? If yes, find one. If no, justify why.

Yes, there are several choices for a circuit, such as

$$(\{1, 4\}, \{4, 7\}, \{7, 5\}, \{5, 2\}, \{2, 3\}, \{3, 6\}, \{6, 7\}, \{7, 2\}, \{2, 1\})$$
$$(\{1, 4\}, \{4, 7\}, \{7, 6\}, \{6, 3\}, \{3, 2\}, \{2, 5\}, \{5, 7\}, \{7, 2\}, \{2, 1\}),$$

and their cyclical representations. ☐

14. (15 points) Let T be a full ternary tree with 50 internal nodes.

Recall that a full ternary tree is one where all nodes have 0 or 3 children. ☐

(a) How many nodes does T have?

T has $1+50 \cdot 3 = 151$ nodes. Initially there is just the root (which is also a leaf). Every time when a leaf becomes an internal node, it gets 3 new children. This happens 50 times. ☐

(b) How many leaves does T have?

Leaves are all those nodes that are not internal. Altogether there are 151 nodes (found in the previous item). We can subtract: $151 - 50 = 101$ leaves. ☐

(c) How many edges does T have?

Any tree has number of edges that is one less than the number of nodes. So the number of edges is $151 - 1 = 150$. ☐

(d) What is the largest and the smallest height of T?

Largest height. We explain why it must be 50.
To get the maximum height, we add new three vertices to the deepest leaf every time. Initially the tree consists of a root alone; the height of the tree is 0. Every time the deepest leaf gets new children, the height increases by one. We can repeat this 50 times.

Smallest height. If we fill all the levels in T to the maximum, we get just $3^0 = 1$ internal node at level 0 (it is the root).
We also get $3^1 = 3$ internal nodes at level 1 (the immediate children of the root).
There are $3^2 = 9$ internal nodes at level 2.
There are 27 internal nodes at level 3.

Since $1+3+9+27 = 40$, there must be also 10 internal nodes at level 4 (namely, the path from the root to any of these nodes has length 4).

Finally, these 10 deepest internal nodes have children (leaves). Their level is larger by 1: $4+1=5$. Therefore the smallest height of such tree is 5. $\square$

15. (15 points) Consider the Boolean expression $x \wedge y \wedge \neg z \vee \neg(x \vee y)$.

(a) Draw the syntax tree of this expression: The leaves of this syntax tree are the propositional variables, the internal nodes are Boolean operations.

Start to analyze the expression by finding the last (lowest-precedence, and the rightmost, if there are many) operator. Write it as a root. Here it is disjunction $\vee$. Then analyze subexpressions in the same manner.

$$\underbrace{x \wedge y \wedge \neg z} \vee \underbrace{\neg(x \vee y)}.$$

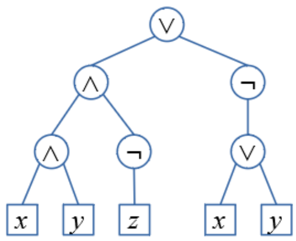


Figure 3: The syntax tree of $x \wedge y \wedge \neg z \vee \neg(x \vee y)$.

$\square$

(b) Write this expression in prefix notation.

Start with the root (write the last-executed operator as the first), after that process all the subexpressions of this root. (In our case the subexpressions are $x \wedge y \wedge \neg z$ and $\neg(x \vee y)$.)

$$\vee\ \wedge\ \wedge\ x\ y\ \neg\ z\ \neg\ \vee\ x\ y.$$

$\square$

(c) Write this expression in postfix notation.

In postfix notation process both subtrees under any node, and only then their parent node:

$$x\ y\ \wedge\ z\, \neg\ \wedge\ x\ y\ \vee\ \neg\ \vee\,.$$

$\square$

Note. Assume that the precedence of Boolean operations is the following: negation, conjunction, disjunction. Both conjunction and disjunction are left associative.