Final

Discrete Structures
April 28, 2021

You must justify all your answers to recieve full credit

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1. (10 points) Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Define a binary relation R on A by

$$R = \left\{ (a, b) \in A \times A \mid a^2 \equiv b^2 \pmod{10} \right\}.$$

   (a) Represent this binary relation as a directed graph.
   (b) Is this relation an equivalence relation on A?
   (c) Is this relation a partial order on A?
   (d) Is this relation a functional relation on A?

2. (10 points) Let $X = \{a, b, c, d, e, f, g\}$. Define a binary relation S on X by

$$S = \{(a, d), (b, e), (c, f), (d, g), (e, a), (f, b), (g, c)\}.$$

   (a) Express the binary relation S as a matrix.
   (b) Express the composition binary relation $S^2 = S \circ S$ as a matrix.
   (c) Find the transitive closure of the binary relation S. How many pairs does the transitive closure of S contain?

3. (10 points) Let n be a positive integer such that its units digit[1] is 3, $n + 1$ is divisible by 3, and $n + 2$ is divisible by 7.

   (a) Write a system of congruences corresponding to the three conditions given on n.
   (b) Find a number n that satisfies these three conditions.
   (c) Find the general solution for the system of congruences from part (a).

4. (10 points) You may use the extended Euclidean algorithm or Blankinship's algorithm for this question.

   (a) Find the inverse of the integer 6 modulo 19.
   (b) Use the inverse from (a) to solve for x in the congruence equation $6x \equiv 7 \pmod{19}$.

5. (10 points) A parking lot with ordered spaces is being built for cars and bicycles. There are two types of bicycles that take up a single parking space each, and three types of cars that take up two parking spaces each. Let p_n be the number of different ways to take up $n \geqslant 1$ parking spaces.

   (a) Find the values p_1 and p_2.
   (b) Find a recurrence relation for p_n.

[1] The "units digit" is the number farthest to the right when n is written in decimal notation.

6. (15 points) Consider the recurrence relation $a_n = 5a_{n-2} + 4a_{n-1}$, with initial conditions $a_0 = 1$ and $a_1 = 2$.

   (a) Give the first 5 terms of the sequence $\{a_n\}_{n=1}^{\infty}$.

   (b) What are the characteristic equation and the characteristic roots of this relation?

   (c) Give the closed form of the solution to this recurrence equation.

7. (15 points) Consider the set $Z = \{1, 2, 3, \ldots, 221\}$.

   (a) How many ways are there to choose 3 elements from Z?

   (b) How many ways are there to choose 3 elements from Z so that none are divisible by 3?

   (c) Suppose you choose n elements from Z. How large must n be so that you are guaranteed to have chosen 3 consecutive elements?

8. (15 points)

   (a) What is the coefficient of x^7y^{11} in the expansion of $(3x - y)^{18}$?

   (b) Consider the permutations $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{smallmatrix}\right)$ and $\rho = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{smallmatrix}\right)$. Find a permutation τ so that $\tau \circ \sigma = \rho$.

9. (15 points) Suppose there are three events, A, B, C, and at least one of them must happen. The probabilities of two of them happening are given in the table below (on the diagonal the probability of each happening is given).

$P(\cdot \cap \cdot)$	A	B	C
A	11/20	8/20	6/20
B	8/20	?	9/20
C	6/20	9/20	11/20

   (a) Are events A and C independent? Why or why not?

   (b) From the table, what is the largest and smallest possible value of $P(A \cap B \cap C)$?

   (c) You are given that $P(A \cap B \cap C) = 5/20$. Draw the Venn diagram representing the three events A, B, C, and compute $P(B)$.

10. (15 points) 2000 people are tested for a disease with a test that gives a yes/no answer[2], and gives it correctly answer 90% of the time. It is known that 20% of the people have the disease. You do not know if you have the disease.

    (a) How many people without the disease will receive an answer saying they have the disease?

    (b) What is the probability that the test says you have the disease?

    (c) Suppose that the test says you have the disease. What is the probability that you actually do have the disease?

[2] "Yes" means you have the disease, "no" means you do not have the disease.

11. (15 points) Neb has a clock that chimes every half hour, chiming as many times as the number on the clock on the hour, and chiming once on the half hour. Neb goes to sleep just after midnight and wakes up just before 7 AM. But he also wakes up twice during the night, and hears a single chime each time. Let X be the random variable representing the time of the second chime.

   (a) Compute $P(X \leqslant 2{:}30 \text{ AM})$. You may assume that waking up at any moment during the night is equally likely.

   (b) Give an expression for the expected value $E(X)$. You do not need to evaluate it.

12. (15 points) Suppose you have two dice, one with 6 sides (having the numbers $1, \ldots, 6$) and one with 8 sides (having the numbers $1, \ldots, 8$). Both dice are fair – their numbers roll with equal probabilities. You roll both at the same time. Let X be the random variable equal to the sum of numbers rolled on both dice.

   (a) Find the variance $V(X)$.

   (b) Use Chebyshev's inequality to estimate the probability $p(X \in \{2, 3, 13, 14\})$.

   (c) Compute the actual probability $p(X \in \{2, 3, 13, 14\})$.

13. (15 points) Consider the following undirected graph G, represented as an adjacency matrix.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

   (a) Draw the graph.

   (b) Is the graph bipartite? If yes, give a bipartition. If no, justify why.

   (c) Does the graph have an Euler circuit? If yes, find one. If no, justify why.

14. (15 points) Let T be a full ternary tree with 50 internal nodes.

   (a) How many nodes does T have?

   (b) How many leaves does T have?

   (c) How many edges does T have?

   (d) What is the largest and the smallest height of T?

15. (15 points) Consider the Boolean expression $x \wedge y \wedge \neg z \vee \neg(x \vee y)$.

   (a) Draw the syntax tree of this expression: The leaves of this syntax tree are the propositional variables, the internal nodes are Boolean operations.

   (b) Write this expression in prefix notation.

   (c) Write this expression in postfix notation.

   Note. Assume that the precedence of Boolean operations is the following: negation, conjunction, disjunction. Both conjunction and disjunction are left associative.