Homework 1

Discrete Structures
Due Tuesday, January 12, 2021

Submit each question separately in .pdf format only

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1. Complete the following truth table.

P	Q	R	$P \wedge Q \wedge R$	$(P \vee Q) \wedge R$	$P \vee (Q \wedge R)$	$P \to (Q \vee R)$	$(P \wedge Q) \leftrightarrow (Q \vee R)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	T	T
T	F	T	F	T	T	T	F
T	F	F	F	F	T	F	T
F	T	T	F	T	T	T	F
F	T	F	F	F	F	T	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	T	T

2. Rewrite the following statements using only symbols (logical quantifiers, connectives, and the symbols $\mathbf{N}, \mathbf{Q}, \mathbf{R}$).

   (a) There is a largest natural number.

   $\exists\, n \in \mathbf{N}\ (\forall\, m \in \mathbf{N}\ (\, m \leqslant n))$ $\square$

   (b) The sum of two rational numbers is a rational number.

   $\forall\, x, y \in \mathbf{Q}\ (x + y \in \mathbf{Q})$ $\square$

   (c) The sum of a rational and an irrational number is a rational number.

   $\forall\, x \in \mathbf{Q}\ (\forall\, y \in \mathbf{R} \setminus \mathbf{Q}\ (x + y \in \mathbf{Q}))$ $\square$

   (d) Between any two rational numbers there is another rational number.

   $\forall\, x, y \in \mathbf{Q}\ ((x < y) \implies (\exists\, z \in \mathbf{Q}\ ((x < z) \wedge (z < y))))$ $\square$

3. Let P, Q, T, F be logical propositions, where T is always true, but F is always false. Prove that the following propositions are tautologies. Indicate every logical equivalence or definition you are using.

(a) $T \wedge (((P \vee (P \wedge Q)) \vee \neg P) \vee (\neg(\neg Q \vee Q)))$

$$
\begin{aligned}
T \wedge ((P \vee \neg P) \vee (\neg(\neg Q \vee Q))) && \text{(absorption law)} \\
(T \wedge (P \vee \neg P)) \vee (T \wedge (\neg(\neg Q \vee Q))) && \text{(distributive law)} \\
(P \vee \neg P) \vee (T \wedge (\neg(\neg Q \vee Q))) && \text{(identity law)} \\
(P \vee \neg P) \vee (\neg(\neg Q \vee Q)) && \text{(identity law)} \\
(P \vee \neg P) \vee (\neg(\neg Q) \wedge (\neg Q)) && \text{(de Morgan's law)} \\
(P \vee \neg P) \vee (Q \wedge (\neg Q)) && \text{(double negation law)} \\
T \vee (Q \wedge (\neg Q)) && \text{(negation law)} \\
T \vee F && \text{(negation law)} \\
T && \text{(domination law)}
\end{aligned}
$$

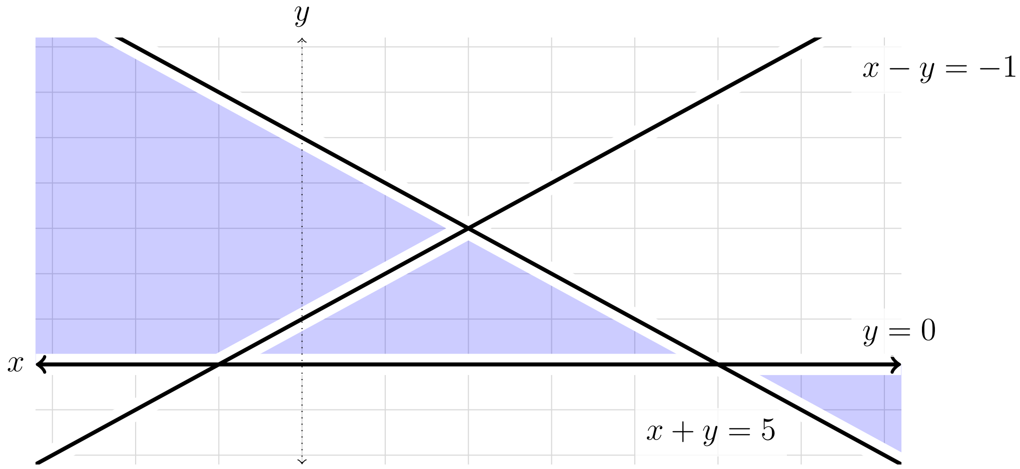
□

(b) $\neg(P \leftrightarrow \neg P)$

$$
\begin{aligned}
\neg((P \implies \neg P) \wedge (\neg P \implies P)) && \text{(definition of } \iff \text{)} \\
\neg((\neg P \vee \neg P) \wedge (\neg P \implies P)) && \text{(definition of } \implies \text{)} \\
\neg((\neg P \vee \neg P) \wedge (\neg(\neg P) \vee P)) && \text{(definition of } \implies \text{)} \\
\neg((\neg P \vee \neg P) \wedge (P \vee P)) && \text{(double negation law)} \\
\neg(((\neg P \vee \neg P) \wedge P) \vee ((\neg P \vee \neg P) \wedge P)) && \text{(distributive law)} \\
\neg((\neg P \vee \neg P) \wedge P) && \text{(idempotent law)} \\
\neg(P \wedge (\neg P \vee \neg P)) && \text{(commutative law)} \\
\neg((P \wedge \neg P) \vee (P \wedge \neg P)) && \text{(distributive law)} \\
\neg(F \vee (P \wedge \neg P)) && \text{(negation law)} \\
\neg(F \vee F) && \text{(negation law)} \\
\neg(F) && \text{(identity law)} \\
T && \text{(definition)}
\end{aligned}
$$

□

4. Consider three lines in the plane, which decompose it into 7 parts.



Let (a, b) be a point in the plane. You are given three propositions about a and b:

- P asserts that $a + b \geqslant 5$
- Q asserts that $a - b \leqslant -1$
- R asserts that $b \geqslant 0$

Using these propositions, answer the questions below.

(a) Write a compound proposition that is true whenever (a, b) belongs to one of the three shaded areas of the diagram. The shaded areas **do not** contain their bounding lines.

We should cover three subcases:
(1) In the shaded pentagonal area to the left: $P = \texttt{False}$, $Q = \texttt{True}$, $R = \texttt{True}$.
(2) In the shaded middle triangle: $P = \texttt{False}$, $Q = \texttt{False}$, $R = \texttt{True}$.
(3) In the shaded triangle to the right: $P = \texttt{True}$, $Q = \texttt{False}$, $R = \texttt{False}$.
In order to check this, pick an arbitrary point in every region and test all three inequalities.
We can write three different conjunctions to express these three areas:
(1) $\neg P \wedge Q \wedge R$,
(2) $\neg P \wedge \neg Q \wedge R$,
(3) $P \wedge \neg Q \wedge \neg R$.
Since any of the (1),(2),(3) can hold, we can build a DNF out of this:

$$\text{Shaded} \equiv (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R).$$

This DNF can be simplified by combining the first two terms:

$$\text{Shaded} \equiv (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \equiv$$

$$\equiv ((\neg P \wedge R) \wedge (Q \vee \neg Q)) \vee (P \wedge \neg Q \wedge \neg R) \equiv$$
$$\equiv (\neg P \wedge R) \vee (P \wedge \neg Q \wedge \neg R).$$

One can drop the variable Q from the last clause as well, since $(P \wedge \neg R)$ (being in the rightmost triangle as in case (3)) implies that the point is also below the line $x - y = -1$. Therefore:

$$\text{Shaded} \equiv (\neg P \wedge R) \vee (P \wedge \neg R) \equiv P \oplus R.$$

You can express this in some other equivalent ways as well; simplification (and getting rid of Q) was not required. □

(b) Write a compound proposition that is logically satisfiable, but no value of $(a, b) \in \mathbf{R}^2$ makes it true.

Answer. The expression $E = P \wedge Q \wedge \neg R$ matches these conditions.
The conjunction $P \wedge Q \wedge \neg R$ is logically satisfiable; the values

$$(P, Q, R) = (\texttt{True}, \texttt{True}, \texttt{False})$$

satisfy it. On the other hand, it is not possible to have all these truth values simultaneously on the picture. Consider some point in the upper area, such as $A(2, 5)$, where $a = 2$, $b = 5$ (see Figure 1). It satisfies conditions P and Q (it is above both the slanted lines: $a + b \geq 5$ and $a - b \leq -1$). Meanwhile, the condition R must also be true (since all the points above both the slanted lines are also above the x axis).

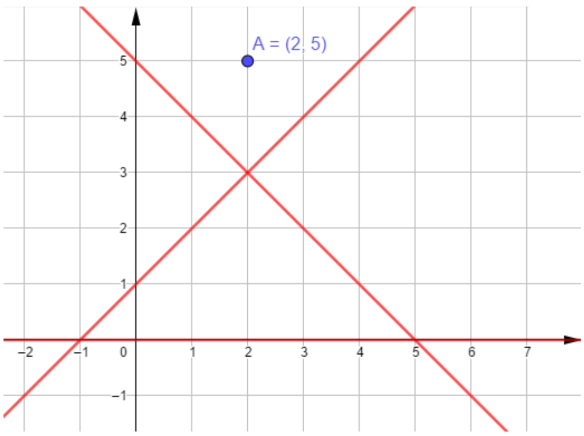


Figure 1: The area containing point $A(2; 5)$ shows that $(P \wedge Q) \rightarrow R$.

Therefore, in the area where the point $A(2; 5)$ is located, we have $(P \wedge Q) \rightarrow R$. But the negation of this implication $(P \wedge Q \wedge \neg R)$ can never be true in this two-dimensional plane. Expression E is the only conjunction of the three values P, Q, R with this property, so any correct answer dependent on these three variables should be equivalent to it. □

Note 1. By $\mathbf{R}^2$ we denote the set of pairs of real numbers; namely, a and b are both real.
Note 2. On a border the truth values of P, Q, R may be indistinguishable from truth values in an adjacent region to the "wrong" side of the line. Everywhere in this problem you can assume that all the points (a, b) only belong to the internal regions, they never fall on any of the three lines.

5. Consider the following proposition, which joins 10 atomic propositions P_i with exclusive or.

$$P_1 \oplus P_2 \oplus P_3 \oplus P_4 \oplus P_5 \oplus P_6 \oplus P_7 \oplus P_8 \oplus P_9 \oplus P_{10} \tag{1}$$

(a) How many terms does its disjunctive normal form (DNF) have? Justify that your DNF contains the smallest possible number of terms.

512 clauses can be used (and that is the smallest number of clauses).
512 clauses are sufficient. There are altogether $2^{10} = 1024$ rows in the truth table of the formula (1); out of these rows exactly half, i.e. $2^9 = 512$ have value `True`. Namely, for any of the 2^9 possible combinations for the truth values of $P_1, P_2, \ldots, P_9$ there exists one value of P_{10} that makes the whole expression (1) true (and the opposite value of P_{10} would make it false).
If we build the full DNF, then every line in the truth table is converted to a term in DNF:

$$\begin{aligned}
(1) \equiv & (\neg P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5 \wedge P_6 \wedge P_7 \wedge P_8 \wedge P_9 \wedge P_{10}) \vee \\
& \vee (\ldots) \vee \cdots \vee (\ldots) \vee \quad \textit{(In all, 10 clauses with one negation)} \\
& \vee (\neg P_1 \wedge \neg P_2 \wedge \neg P_3 \wedge P_4 \wedge P_5 \wedge P_6 \wedge P_7 \wedge P_8 \wedge P_9 \wedge P_{10}) \vee \\
& \vee (\ldots) \vee \cdots \vee (\ldots) \vee \quad \textit{(In all, 120 clauses with three negations)} \\
& \vee (\neg P_1 \wedge \neg P_2 \wedge \neg P_3 \wedge \neg P_4 \wedge \neg P_5 \wedge P_6 \wedge P_7 \wedge P_8 \wedge P_9 \wedge P_{10}) \vee \\
& \vee (\ldots) \vee \cdots \vee (\ldots) \vee \quad \textit{(In all, 252 clauses with five negations)} \\
& \vee (\neg P_1 \wedge \neg P_2 \wedge \neg P_3 \wedge \neg P_4 \wedge \neg P_5 \wedge \neg P_6 \wedge \neg P_7 \wedge P_8 \wedge P_9 \wedge P_{10}) \vee \\
& \vee (\ldots) \vee \cdots \vee (\ldots) \vee \quad \textit{(In all, 120 clauses with seven negations)} \\
& \vee (\neg P_1 \wedge \neg P_2 \wedge \neg P_3 \wedge \neg P_4 \wedge \neg P_5 \wedge \neg P_6 \wedge \neg P_7 \wedge \neg P_8 \wedge \neg P_9 \wedge P_{10}) \vee \\
& \vee (\ldots) \vee \cdots \vee (\ldots). \quad \textit{(In all, 10 clauses with nine negations)}
\end{aligned} \tag{2}$$

Every clause in this long formula contains all 10 variables; but some odd number (one, three, five, seven or nine) variables have negation applied; and you apply them in all possible combinations. We get $10 + 120 + 252 + 120 + 10 = 512$ clauses.

512 clauses are necessary. It is not possible to write the DNF with fewer than 512 clauses. The full DNF has every term containing all 10 variables; therefore it is true just for one of the 2^{10} rows. Writing terms with fewer variables would allow to cover more rows. For example,

$$\ldots \vee (P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5 \wedge P_6 \wedge P_7 \wedge P_8 \wedge P_9) \vee \ldots$$

would be true for two rows (as it does not depend on P_{10} at all). But such expression cannot compute (1), since the value of any value P_{10} can be flipped from `True` to `False` or back, and it should change the value of the XORed expression, but here it does not. Therefore, we can only use terms with all 10 variables, and we need at least 512 of them – one per every value `True` in the truth table.
Note. Not a part of the exercise, but the total number of connectors ($\vee$,$\wedge$,$\neg$) is even larger than the number of terms:

$$511 + 10 \cdot 10 + 120 \cdot 12 + 252 \cdot 14 + 120 \cdot 16 + 10 \cdot 18 = 7679. \tag{3}$$

In this expression 511 is the total number of disjunctions between the 512 terms, there are 10 terms using 10 connectors each (nine conjunctions, one negation), 120 terms using 12 connectors each, 252 terms using 14 connectors each, 120 terms using 16 connectors each and 10 terms using 18 connectors each.

□

(b) How many terms does its conjunctive normal form (CNF) have?

Answer: 512 clauses can be used (and that is the smallest number of clauses). Similar to the previous case – it is sufficient to have 512 terms, since the full CNF has one term per every row in the truth table that is `False`. There are $1024 - 512 = 512$ such rows.

One cannot do better; the number of clauses cannot be decreased, because we can argue that every clause $(P_1 \wedge P_2 \wedge \ldots P_{10})$ or similar (with some negations added) should contain all 10 variables; otherwise some variable is missing. The CNF would become insensitive to the flipping of that variable, which is wrong for the XOR operation. □

(c) Write the equation (1) using as few logical connectors ($\vee$, $\wedge$, $\neg$) as possible. You may use the connectors $\vee$, $\wedge$, $\neg$, and parentheses in any way, but your statement does not need to be in CNF or in DNF. Explain, why your expression is logically equivalent to (1).

Answer: It is possible to compute (1) using either 36 connectors (with some cheating) or 140 connectors (without cheating). There is no claim that one cannot do better than that. Both cases are a big improvement over the full DNF; according to (3) it would need 7679 connectors.

Case 1. In practice one would build the long XOR expression using 36 connectors AND, OR, NOT (and reusing some intermediary results). You can express a single XOR operation using just 4 connectors (two $\wedge$, one $\vee$, one $\neg$):

$$X \oplus Y \equiv (X \vee Y) \wedge \neg(X \wedge Y)$$

The formula says that either X or Y is true, but they cannot both be true. Since (1) contains 9 operations $\oplus$, you can express it using $9 \cdot 4 = 36$ connectors, namely, you need to evaluate some Boolean AND, OR, NOT just 36 times (plus save the intermediary results like $P_1 \oplus P_2$, $P_1 \oplus P_2 \oplus P_3$ etc. in variables). Figure 2 shows how to compute $P_1 \oplus P_2 \oplus P_3 \oplus P_4$ using just 12 Boolean gates ($\wedge$, $\vee$, $\neg$); this "cascade" can be continued until all $P_1, \ldots, P_{10}$ are connected with the XOR operation $\oplus$. Yet, this circuit includes cheating: Subexpressions such as $P_1 \oplus P_2$ are copied and reused. (Red circles mark the locations where this cheating happens.)

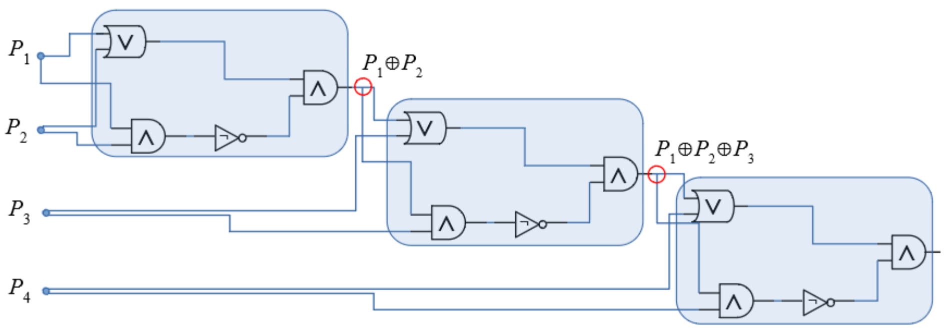


Figure 2: Boolean Circuit to Compute $((P_1 \oplus P_2) \oplus P_3) \oplus P_4$.

Case 2. If you need to create one big formula (and cannot "reuse" its subexpressions), then you need more connectives, because you need to rebuild subexpressions like $(P_1 \oplus P_2)$ or $(P_1 \oplus P_2 \oplus P_3)$ (and longer ones) multiple times. Let us start gradually with short XOR expressions for 2 and 3 variables respectively:

$$\begin{cases} X \oplus Y \equiv (X \vee Y) \wedge \neg(X \wedge Y), \\ X \oplus Y \oplus Z \equiv (X \wedge Y \wedge Z) \vee ((X \vee Y \vee Z) \wedge \neg((X \wedge (Y \vee Z)) \vee (Y \wedge Z))). \end{cases}$$

The above formula was discussed above; the formula for $X \oplus Y \oplus Z$ basically tells that this XOR is true, iff either all X, Y, Z are true, or at least one of them is true (but two cannot be true). Formula for $X \oplus Y$ uses 4 connectives and the formula for

$X \oplus Y \oplus Z$ uses 11 connectives. We will use the associativity of XOR and rewrite the expression (1) like this:

$$((P_1 \oplus P_2) \oplus (P_3 \oplus P_4 \oplus P_5)) \oplus ((P_6 \oplus P_7) \oplus (P_8 \oplus P_9 \oplus P_{10})).$$

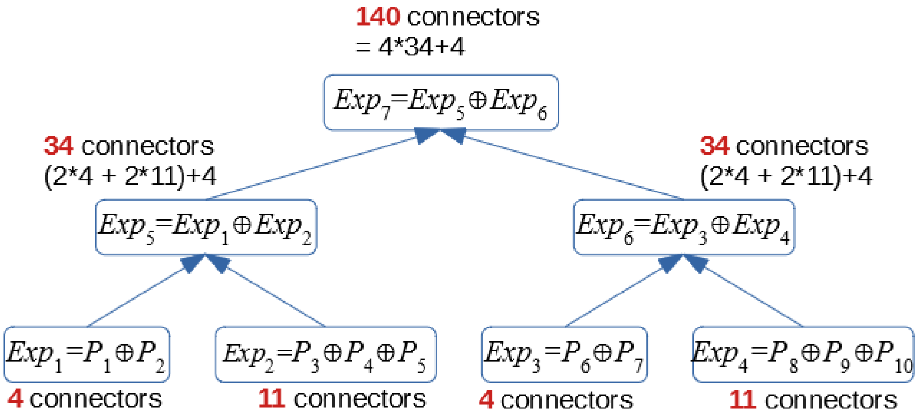


Figure 3: Counting connectors as we move up the hierarchy.

Once we know how many operations does it take to create XOR of 2 or 3 variables, we can combine them. For 5 variables we need to re-apply the formula to calculate the XOR for two variables:

$$Expr_5 \;\equiv\; Expr_1 \oplus Expr_2 \;\equiv\; ((Expr_1 \vee Expr_2) \wedge \neg(Expr_1 \wedge Expr_2)).$$

It repeats $Expr_1$ twice and $Expr_2$ twice and it also introduces four new connectors. So, the total number of connectors used is $2 \cdot 4 + 2 \cdot 11 + 4 = 34$. Similarly, Exp_6 also needs 34 connectors.

As we combine $Expr_5$ and $Expr_6$ into the final expression $Expr_7$, each of the subexpressions is used twice, and there are four new connectors, which brings their total number to

$$2 \cdot 34 + 2 \cdot 34 + 4 = 140.$$

Note. In order to get full credit for this 5(C), you do not need to analyze both interpretations (or match/improve the estimates written here). Any consistent interpretation and correct answer (some reasonable estimate for the minumum number of Boolean connectors) would be fine.

□

Note. A "term" in a DNF is any conjunction of P_i or their negations participating in the long disjunction. For example, the following DNF:

$$(A \wedge \neg B \wedge \neg C) \vee (\neg D \wedge E) \vee C \vee (A \wedge \neg E)$$

has 4 terms: $(A \wedge \neg B \wedge \neg C)$, $(\neg D \wedge E)$, (C) and $(A \wedge \neg E)$.