Homework 2

Discrete Structures
Due Tuesday, January 19, 2021

Submit each question separately in .pdf format only

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1. Let $P(x)$ be the statement "x is a perfect square" and $Q(x)$ be the statement "three times x is a perfect square".

   (a) Write the following quantifications as sentences in English:

      i. $\forall\ x \in \mathbf{Z}\ (\forall\ y \in \mathbf{Z}\ ((Q(x) \vee P(x))\ \leftrightarrow\ (x - y > 0))$

      For all integers x, y, $x - y$ is positive if and only if x is a perfect square or $3x$ is a perfect square. □

      ii. $\exists\ x \in \mathbf{Z}\ (\exists\ y \in \mathbf{Z}\ (\forall\ z \in \mathbf{Z}\ ((x \neq y)\ \wedge (P(x)\ \rightarrow\ Q(y + z)))))$

      For all integers x, y, z with x different from y, if x is a perfect square, then $3y + 3z$ is a perfect square. □

   (b) Write the following sentences in English as quantifications:

      i. Every integer is one less or two more than a perfect square.

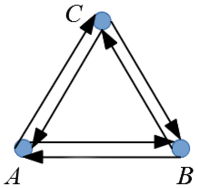
      $\forall\ x \in \mathbf{Z}\ (P(x+1)\ \vee\ P(x-2))$ □

      ii. It is never the case that a perfect square is six times a different perfect square.

      $\neg\,(\exists\ x \in \mathbf{Z}\ (\exists\ y \in \mathbf{Z}\ (P(x)\ \wedge\ (x = 6y)\ \wedge\ P(y)\ \wedge (x \neq y))))$ □

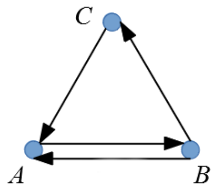
2. For a set of three cities define predicates $Plane(x, y)$ and $Rail(x, y)$ that are true iff there is a direct link by plane or rail, respectively, from city x to city y. These are represented by tables and diagrams below.

$Plane(x, y)$:



$x \backslash y$	A	B	C
A	F	T	T
B	T	F	T
C	T	T	F

$Rail(x, y)$:



$x \backslash y$	A	B	C
A	F	T	F
B	T	F	T
C	T	F	F

Write the following Boolean propositions with quantifiers and justify, why these statements are true or false.

   (a) From any city there is a direct plane-link to some other city.

   Let $S = \{A, B, C\}$ be the set of all cities. Omitting this set does not cause ambiguity in this question, but it looks neater, if quantifiers specify their domain set.
   Proposition: $\forall\ x \in S\ \exists\ y \in S\ ((x \neq y)\ \wedge\ Plane(x, y))$.
   Truth value: This statement is true, because every row in the *Plane* table above has at least one `T`. □

(b) From any city one can go to any other city in two steps like this: First take a plane-link and then take a rail-link.

Proposition: $\forall\, x \in S\ \forall\, y \in S\ \exists\, z \in S\ ((x \neq y) \ \rightarrow\ (Plane(x,z)\ \wedge\ Rail(z,y)))$.
Truth value: This statement is false. Assume that your source city (x in the above formula) is B, and your destination city is $y = C$. Then you would need to have the intermediate city $z = B$ (the only rail-link into C is from B), but there is no plane-link from the city B to itself. □

(c) No matter what are the cities, if it is possible to go from a city x to some other city y with two plane-links, then it is also possible to go from x to y using a single plane-link.

Proposition: $\forall\, x \in S\ \forall\, z \in S\ \forall\, y \in S\ (Plane(x,z)\ \wedge Plane(z,y)\ \rightarrow\ Plane(x,y))$.
Truth value: This statement is false. One can go from $x = A$ to $y = A$ with two plane-links (for example, from A to B and then from B to A), but one cannot go from A to A with just one link: There is no direct plane connection from A to itself.
Note 1. A two-argument predicate with this property is called a *transitive relation.* For example, if $u \leq v$ and $v \leq w$, then also $u \leq w$; and we say that the relation $\leq$ is transitive.
Note 2. If you interpreted the problem in such a way that x and y must be two **different** cities, and had this precondition $(x \neg y)$ in your predicate statement, then the statement becomes true, since you can travel between any two different cities via the third city (and also can travel directly).

□

3. Consider the following sets.

- Let $\mathbf{R}^3$ be the set of all points in a three-dimensional space. That is, any point $A = A(x_A, y_A, z_A) \in \mathbf{R}^3$ in this set has three real coordinates $x_A, y_A, z_A \in \mathbf{R}$.
- Let P be the set of all two-dimensional planes in $\mathbf{R}^3$.

Let $A, B \in \mathbf{R}^3$ and $\alpha, \beta \in P$. Consider the following predicates.

- $S(A, \alpha)$: "the plane α goes through the point A", or equivalently, "the point A lies in the plane α"
- $I(A, B)$: "the points A and B are the same"
- $I(\alpha, \beta)$: "the planes α and β are the same"

Using only these sets and predicates, express the following new predicates and Boolean propositions.

(a) Predicate $U(A, B, C, \alpha)$: The plane α goes through the points $A, B, C \in \mathbf{R}^3$.

Translation: $U(A, B, C, \alpha)\ :=\ S(A, \alpha) \wedge S(B, \alpha) \wedge S(C, \alpha)$. □

(b) Predicate $V(\alpha, \beta)$: The planes α and β are parallel. That is, they do not share any points.

Translation: $V(\alpha,\beta) \;:=\; \forall A \in \mathbf{R}^3 \; (\neg S(A,\alpha) \;\vee\; \neg S(A,\beta))$.
(By De Morgan's law it is the same as $\neg\exists A \in \mathbf{R}^3 \; (S(A,\alpha) \;\wedge\; S(A,\beta))$. □

(c) Predicate $W(A,B,C)$: The points $A,B,C \in \mathbf{R}^3$ are on the same line.

Translation:

$$W(A,B,C) \;:=\; \exists\alpha \in P \; \exists\beta \in P \; (\neg I(\alpha,\beta) \wedge$$
$$\wedge \; (S(A,\alpha) \wedge S(B,\alpha) \wedge S(C,\alpha)) \wedge \; (S(A,\beta) \wedge S(B,\beta) \wedge S(C,\beta))).$$

Or use a predicate $U(A,B,C,\alpha)$ defined above:

$$W(A,B,C) \;:=\; \exists\alpha \in P \; \exists\beta \in P \; (\neg I(\alpha,\beta) \;\wedge\; U(A,B,C,\alpha) \;\wedge\; U(A,B,C,\beta)).$$

Note. In other words, there are two different planes containing all three points. □

(d) Proposition Pr_1: There exist four points in $\mathbf{R}^3$ such that no plane goes through them all.

Translation: $\mathrm{Pr}_1 \;:=$

$$\exists A,B,C,D \in \mathbf{R}^3 \; \forall\alpha \in P \; (\neg S(A,\alpha) \vee \neg S(B,\alpha) \vee \neg S(C,\alpha) \vee \neg S(D,\alpha)).$$

□

(e) Proposition Pr_2: For any three points in $\mathbf{R}^3$, there exist three planes such that all are parallel, the first plane goes through the first point, the second plane goes through the second point, and the third plane goes through the third point.

Translation: $\mathrm{Pr}_2 \;:=$

$$\forall A,B,C \in \mathbf{R}^3 \; \exists\alpha,\beta,\gamma \in P \; (V(\alpha,\beta) \wedge V(\alpha,\gamma) \wedge (S(A,\alpha) \wedge S(B,\beta) \wedge S(C,\gamma))).$$

□

You may use 2 predicates defined above (and also predicates U,V,W, if they are already defined in earlier steps). You may also use Boolean connectors $\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$ and quantifiers. Be sure to indicate over which set the quantifiers operate.

4. Let $\triangle ABC$ be a triangle in the plane, with vertices $A = A(x_A, y_A)$, $B = B(x_B, y_B)$, and $C = C(x_C, y_C)$.

   (a) Suppose that $\triangle ABC$ is equilateral and that A, B have integer coordinates (that is, $x_A, y_A, x_B, y_B \in \mathbf{Z}$). Prove that the area of $\triangle ABC$ is an irrational number.

Proof: Denote the length of the side $|AB|$ by a. Since x_A, y_A, x_B, y_B are all integers,

$$|AB|^2 = (x_A - x_B)^2 + (y_A - y_B)^2 \in \mathbf{Z}.$$

The equality holds by the Pythagorean theorem. We also know that an area of an equilateral triangle with side a is

$$S_{\triangle ABC} = \frac{1}{2}a^2 \sin 60^\circ = \frac{a^2\sqrt{3}}{4}. \tag{1}$$

Claim #1: $\sqrt{3}$ is an irrational number.
Assume by contradiction that $\sqrt{3} = \frac{p}{q}$, which is an irreducible fraction for some positive integers p, q. Then, after squaring both sides, $3q^2 = p^2$ and we get that p^2 is divisible by 3. Therefore p itself is divisible by 3 and can express $p = 3k$ for some integer k. Then $3q^2 = (3k)^2 = 9k^2$ or $q^2 = 3k^2$. We also get that q^2 is divisible by 3. Thus both p and q are divisible by 3; it contradicts the assumption that there is an irreducible fraction equal to $\sqrt{3}$.
(Claim #1 is proven.)

Claim #2: The product of a rational number $r \neq 0$ and an irrational number α is always irrational.
Assume by contradiction that $r \cdot \alpha = r_1$, where r_1 is a rational number. In this case we can express $\alpha = r_1/r$, and it would be rational (as a fraction of two rational numbers). It contradicts the assumption that α is irrational.
(Claim #2 is proven.)

Return to the equation (1). $S_{\triangle ABC}$ equals the product of a rational number $\frac{a^2}{4} = \frac{(x_A - x_B)^2 + (y_A - y_B)^2}{4}$ and an irrational number $\sqrt{3}$ (by Claim #1). Their product is irrational (by Claim #2). $\square$

(b) Suppose that A, B, C have integer coordinates. Prove that the area of $\triangle ABC$ is a rational number.

Proof: Let $\triangle ABC$ have all integer coordinates. Use Pick's theorem: since ABC is a simple polygon (no sides intersecting other sides), its area is $S_{\triangle ABC} = i + \frac{b}{2} - 1$, where i is the number of interior points and b is the number of boundary points. From here we conclude that the area is even an integer number (or a half of an integer number). So it must be a rational number.
If you do not want to use Pick's theorem, draw a bounding rectangle around the triangle ABC using the gridlines. It has integer area, and $S_{\triangle ABC}$ can be obtained by subtracting two or three orthogonal triangles with integer (or half-integer) areas. Figure 1 shows both methods to compute the area. By Pick's theorem

$$S_{\triangle ABC} = 16 + \frac{4}{2} - 1 = 17.$$

By subtraction of gray triangles:

$$S_{\triangle ABC} = S_{CKLM} - S_{CKA} - S_{ALB} - S_{BMC} = 42 - \frac{7 \cdot 4}{2} - \frac{2 \cdot 5}{2} - \frac{6 \cdot 2}{2} = 17.$$

$\square$

(c) Is it possible for $\triangle ABC$ to be equilateral triangle and for A, B, C to have rational coordinates? Find an example of such a triangle or prove that no such triangle exists.

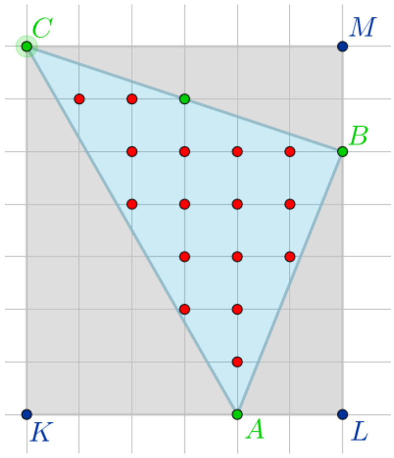


Figure 1: Triangle ABC with all integer vertices.

Statement. It is not possible to draw an equilateral triangle ABC with all rational coordinates.

Proof. Assume that an equilateral $\triangle ABC$ has all six coordinates x_A, y_A, x_B, y_B, x_C, y_C in $\mathbf{Q}$ (all coordinates are rational). All six numbers have denominators; denote them by $q_1, q_2, q_3, q_4, q_5, q_6$. Multiply all the coordinates by the product $q_1 \cdot q_2 \cdot q_3 \cdot q_4 \cdot q_5 \cdot q_6$. After this manipulation, the triangle is uniformly scaled (it is still equilateral triangle) and the coordinates for A, B, C are now integers.

By **(a)** the triangle $\triangle ABC$ has irrational area (it is equilateral, and the square of its side a^2 is an integer). By **(b)** the triangle $\triangle ABC$ has rational area (all vertices are integers). The number $S_{\triangle ABC}$ cannot be rational and irrational at the same time. This is a contradiction.

Note. This note refers to some knowledge that was not covered in the textbook and in the class (until now), but it will be important as we move on to Chapter 2.6 (matrices). This problem shows that there are two very different and incompatible alternatives:

Alternative 1. One can certainly draw equilateral triangles on a grid paper. But in (a) it is shown that such a triangle (if two vertices A, B have integer coordinates) will have irrational area and the third vertex will also be irrational. See Figure 2 – it shows points $A(0;0)$ and $B(15,4)$, and the third point C obtained by rotating the line segment AB counterclockwise by 60°. Since we are rotating around the origin $A(0;0)$, the coordinates of C can be obtained, using *Rotation matrix* (see `https://bit.ly/3pc1c3F`):

$$\left(\begin{array}{c} x_C \\ y_C \end{array} \right) = \left(\begin{array}{cc} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{array} \right) \cdot \left(\begin{array}{c} x_B \\ y_B \end{array} \right).$$

Let us carry out this calculation and use values $x_B = 15$, $y_B = 4$, and $\alpha = 60^\circ = \frac{\pi}{3}$.

$$\left\{ \begin{array}{l} x_C = x_B \cdot \cos\alpha - y_B \cdot \sin\alpha = 15 \cdot \frac{1}{2} - 4 \cdot \frac{\sqrt{3}}{2} = 4.035898384862247\ldots, \\ y_C = x_B \cdot \sin\alpha + y_B \cdot \cos\alpha = 15 \cdot \frac{\sqrt{3}}{2} + 4 \cdot \frac{1}{2} = 14.990381056766578\ldots. \end{array} \right.$$

In this case C has irrational coordinates (they can be made close to integer values, but they will never match integers exactly). And the area of the triangle $\triangle ABC$ is irrational as well.

Alternative 2. One can draw a triangle $\triangle ABC$ with vertices located exactly in the grid intersections (all integer coordinates). In this case the area is easy to compute; it is always an integer (or an integer plus $\frac{1}{2}$). Furthermore, it can be very close to an equilateral triangle, but it will never be exactly equilateral one.

$\square$

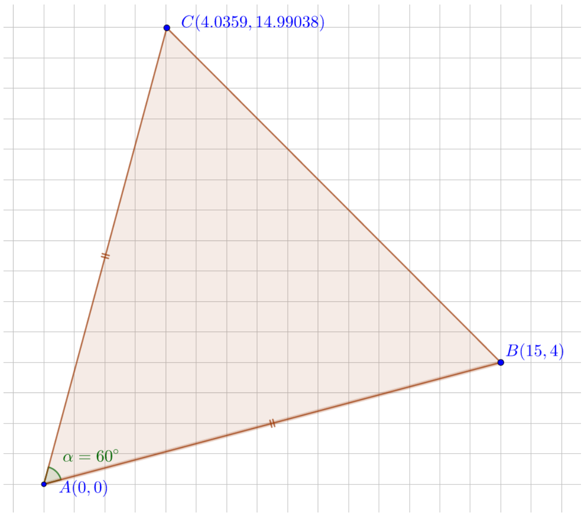


Figure 2: Equilateral $\triangle ABC$ with irrational coordinates C (rounded to 5 places).

Hint: You may use Pick's theorem for part (b): `https://bit.ly/39m3qXH`.

5. Find your Student ID in ORTUS or ask the instructor(s). It has format similar to this: `201RDB999`, but the last three digits may be different. Just extract the last three digits and we denote this number by $\overline{abc}$. In our example $\overline{abc} = 999$. Then find the expression $N = \overline{abc}\%30 + 1$ (the remainder when dividing this number by 30 plus 1). In our case $N = 9 + 1 = 10$. After that search what are the tautologies posted by "mathslogicrobot" for December N, and take the top tautology from the list. (If the bot did not tweet any tautology on that day, take the following day.)

   In our example, take December 10, 2020. Visit the Twitter website: `https://twitter.com/`; enter the search string to find all the results between "since" and "until" (see Figure 3).

Figure 3: Finding a Tautology from Dec 10.

Add the `forall` quantifier and prove the corresponding Lemma in Coq. Submit a file named `tautology.v` as the solution for your Problem 5. In our example, the tautology you have to prove is this (see Figure 4.)

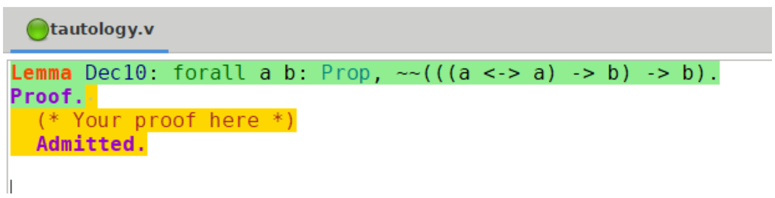


Figure 4: Coq IDE Screenshot.

Your proof can import a classical logic axiom, if necessary. But the proof should not use "Admitted" or "tauto" or any other trivial method. Instead, your proof should be a valid step by step application of Coq tactics from "Proof" to "Qed".

Sample Answer. There are about 30 different tautologies (depending on your Student ID; all posted during December, 2020). Visit `https://bit.ly/35ZsvXx` to display the list of tautologies posted on January 1. We can discuss all of them. Below is the proof of the first tautology posted in the year 2021:

$$\forall a \in \texttt{Prop}\ \forall b \in \texttt{Prop}\ (\neg\neg((a \vee (b \leftrightarrow a)) \vee b)).$$

In Coq `Prop` denotes the set of all propositions taking values `True` or `False`. The proof below shows the "brute-force" proof – simply sort the cases when variables b and a are true or false (using the Excluded Middle axiom from `Classical_Prop` named `classic`).

There could be more "beautiful" proofs (not using case-by case sorting too much), but for them we would need additional lemmas, such as regrouping items in a disjunction ($\vee$) or transforming disjunctions into implications. □

File `tautology.v`

```
Require Import Classical_Prop.

Lemma Jan1: forall a b: Prop, ~~((a \/ (b <-> a)) \/ b).
Proof.
  intros a b.
  assert ((a \/ (b <-> a)) \/ b) as H.
  pose (classic b) as H2.
  destruct H2 as [bTrue | bFalse].
  right.
  exact bTrue.
  left.
  pose (classic a) as H3.
  destruct H3 as [aTrue | aFalse].
  left.
  exact aTrue.
  right.
  split.
  intros bTrue.
  contradiction (bFalse bTrue).
  intros aTrue.
  contradiction (aFalse aTrue).
  unfold not.
  intros H4.
  contradiction (H4 H).
Qed.
```