Homework 2

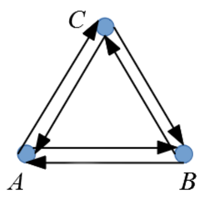
Discrete Structures
Due Tuesday, January 19, 2021

Submit each question separately in .pdf format only

1. Let $P(x)$ be the statement "x is a perfect square" and $Q(x)$ be the statement "three times x is a perfect square".
   (a) Write the following quantifications as sentences in English:
      i. $\forall\, x \in \mathbf{Z}\ (\forall\, y \in \mathbf{Z}\ ((Q(x) \vee P(x)) \ \leftrightarrow\ (x - y > 0))$
      ii. $\exists\, x \in \mathbf{Z}\ (\exists\, y \in \mathbf{Z}\ (\forall\, z \in \mathbf{Z}\ ((x \neq y)\ \wedge (P(x)\ \rightarrow\ Q(y + z)))))$
   (b) Write the following sentences in English as quantifications:
      i. Every integer is one less or two more than a perfect square.
      ii. It is never the case that a perfect square is six times a different perfect square.

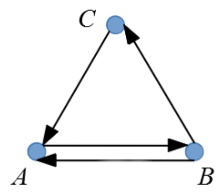
2. For a set of three cities define predicates $Plane(x, y)$ and $Rail(x, y)$ that are true iff there is a direct link by plane or rail, respectively, from city x to city y. These are represented by tables and diagrams below.

$Plane(x, y)$:



x \ y	A	B	C
A	F	T	T
B	T	F	T
C	T	T	F

$Rail(x, y)$:



x \ y	A	B	C
A	F	T	F
B	T	F	T
C	T	F	F

Write the following Boolean propositions with quantifiers and justify, why these statements are true or false.

   (a) From any city there is a direct plane-link to some other city.
   (b) From any city one can go to any other city in two steps like this: First take a plane-link and then take a rail-link.
   (c) No matter what are the cities, if it is possible to go from a city x to some other city y with two plane-links, then it is also possible to go from x to y using a single plane-link.

3. Consider the following sets.

   - Let $\mathbf{R}^3$ be the set of all points in a three-dimensional space. That is, any point $A = A(x_A, y_A, z_A) \in \mathbf{R}^3$ in this set has three real coordinates $x_A, y_A, z_A \in \mathbf{R}$.
   - Let P be the set of all two-dimensional planes in $\mathbf{R}^3$.

   Let $A, B \in \mathbf{R}^3$ and $\alpha, \beta \in P$. Consider the following predicates.

   - $S(A, \alpha)$: "the plane α goes through the point A", or equivalently, "the point A lies in the plane α"
   - $I(A, B)$: "the points A and B are the same"
   - $I(\alpha, \beta)$: "the planes α and β are the same"

   Using only these sets and predicates, express the following new predicates and Boolean propositions.

   (a) Predicate $U(A, B, C, \alpha)$: The plane α goes through the points $A, B, C \in \mathbf{R}^3$.

   (b) Predicate $V(\alpha, \beta)$: The planes α and β are parallel. That is, they do not share any points.

   (c) Predicate $W(A, B, C)$: The points $A, B, C \in \mathbf{R}^3$ are on the same line.

   (d) Proposition Pr_1: There exist four points in $\mathbf{R}^3$ such that no plane goes through them all.

   (e) Proposition Pr_2: For any three points in $\mathbf{R}^3$, there exist three planes such that all are parallel, the first plane goes through the first point, the second plane goes through the second point, and the third plane goes through the third point.

   You may use 2 predicates defined above (and also predicates U, V, W, if they are already defined in earlier steps). You may also use Boolean connectors $\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$ and quantifiers. Be sure to indicate over which set the quantifiers operate.

4. Let $\triangle ABC$ be a triangle in the plane, with vertices $A = A(x_A, y_A)$, $B = B(x_B, y_B)$, and $C = C(x_C, y_C)$.

   (a) Suppose that $\triangle ABC$ is equilateral and that A, B have integer coordinates (that is, $x_A, y_A, x_B, y_B \in \mathbf{Z}$). Prove that the area of $\triangle ABC$ is an irrational number.

   (b) Suppose that A, B, C have integer coordinates. Prove that the area of $\triangle ABC$ is a rational number.

   (c) Is it possible for $\triangle ABC$ to be equilateral triangle and for A, B, C to have rational coordinates? Find an example of such a triangle or prove that no such triangle exists.

   Hint: You may use Pick's theorem for part (b): `https://bit.ly/39m3qXH`.

5. Find your Student ID in ORTUS or ask the instructor(s). It has format similar to this: `201RDB999`, but the last three digits may be different. Just extract the last three digits and we denote this number by $\overline{abc}$. In our example $\overline{abc} = 999$. Then find the expression $N = \overline{abc}\%30 + 1$ (the remainder when dividing this number by 30 plus 1). In our case $N = 9 + 1 = 10$. After that search what are the tautologies posted by "mathslogicrobot" for December N, and take the top tautology from the list. (If the bot did not tweet any tautology on that day, take the following day.)

   In our example, take December 10, 2020. Visit the Twitter website: `https://twitter.com/`; enter the search string to find all the results between "since" and "until" (see Figure 1).

Figure 1: Finding a Tautology from Dec 10.

   Add the `forall` quantifier and prove the corresponding Lemma in Coq. Submit a file named `tautology.v` as the solution for your Problem 5. In our example, the tautology you have to prove is this (see Figure 2.)

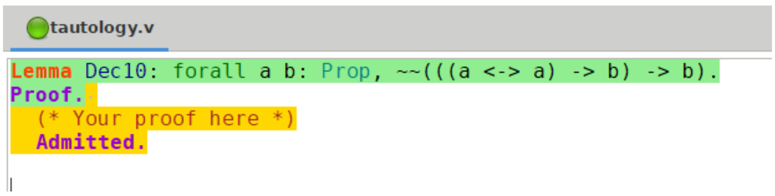


Figure 2: Coq IDE Screenshot.

   Your proof can import a classical logic axiom, if necessary. But the proof should not use "Admitted" or "tauto" or any other trivial method. Instead, your proof should be a valid step by step application of Coq tactics from "Proof" to "Qed".