# Homework 3

Discrete Structures
Due Tuesday, January 26, 2021

**Submit each question separately in .pdf format only**

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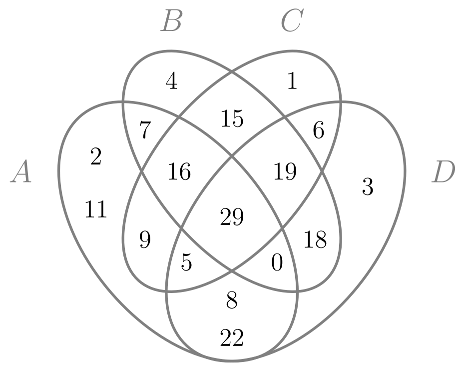
1. (a) Given the Venn diagram on the left, write all the elements of the sets on the right.




$(A \cap C) \setminus (D \cap B)$

$\overline{(\overline{A} \cup D)} \cap B$

$(C \oplus D) \cap \overline{(D \oplus B)}$

$\overline{(D \cap B \cap C)} \oplus (D \cup B \cup C)$

(b) Describe the following sets using $A, B, C, D$ from above and set operations on them.

$$E = \{16, 29, 0\} \qquad F = \{6, 7, 16, 19\} \qquad G = \{3, 18, 0, 4, 7\}$$

(c) Simplify the following sets as much as possible. That is, rewrite them without using the union $\cup$ or intersection $\cap$ symbols.

$$X = \bigcup_{i=0}^{\infty} [i, i+1] \qquad Y = \bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right] \qquad Z = \bigcap_{n=1}^{\infty} \left\{\frac{n}{x} \;:\; x \in \mathbf{Z}_{\geqslant n}\right\}$$

2. Let $A, B, C$ be sets, and let $f\colon A \to B$ and $g\colon B \to C$ be functions.

   (a) Using logical symbols, express the following statements.

   i. $g$ is injective when restricted to the range of $f$

   ii. there exists an element in $C$ whose preimage in $g$ is not $f(a)$ for any $a$ in $A$

   (b) If $f$ and $g$ are injective, prove that $g \circ f$ is injective.

   (c) If $g \circ f$ is surjective, prove that $g$ must be surjective.

3. Let $A, B, C$ be arbitrary sets in the same universe $U$. Prove or disprove the following statements:

   (a) $(B \cup C) - A = (B - C) \cup (C - A)$.

   (b) $(B \oplus C) - A = (B - A) \oplus (C - A)$.

   (c) $\overline{A} \times \overline{(B \cup C)} = \overline{A \times (B \cup C)}$.

4. Prove or disprove the following statements about power sets.

   (a) There is a set $X$ such that its powerset $\mathcal{P}(X)$ equals

$$\{\varnothing, \{a\}, \{\varnothing\}, \{a, \{\varnothing\}\}\}. \tag{1}$$

(b) There is a set $X$ such that its powerset $\mathcal{P}(X)$ equals

$$\{\varnothing, \{\varnothing\}, \{\{a\}\}, \{\{a,b\}\}, \{\varnothing, \{a\}\}, \{\varnothing, \{a,b\}\}, \{\{a\}, \{a,b\}\}, \{\varnothing, \{a\}, \{a,b\}\}\}. \quad (2)$$

(c) For any two sets $A$ and $B$, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ iff $A \subseteq B$.

5. Prove the following three tautologies using Coq. Submit your file `tautology.v` as the solution for your Problem 5.

```
Lemma Sample5A: forall P Q:Prop, ~(~P /\ ~Q) -> P \/ Q.
Proof.
  (* Place your proof here *)
Qed.

Lemma Sample5B: forall P Q:Prop, (P -> Q) -> (~P \/ Q).
Proof.
  (* Place your proof here *)
Qed.

Lemma Sample5C: forall P Q:Prop, (P -> Q) <-> (~Q -> ~P).
Proof.
  (* Place your proof here *)
Qed.
```

*Note.* Most lemmas in the non-constructive mathematics are proven using some tautology as an axiom. Either the “NNPP axiom” ($\neg\neg A \to A$, double negation elimination) or the “classic axiom” ($A \vee \neg A$, the law of the Excluded Middle). See the link *Week3 > Two Nonconstructive Proofs of the Same Lemma* in ORTUS. You can try out whichever method you want. For these axioms to work the first line in your proof should be:
`Require Import Classical_Prop.`