Homework 4

Discrete Structures
Due Tuesday, February 2, 2021

Submit each question separately in .pdf format only

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1. Consider the sets $X_{1,1} = \{1\}$, $X_{2,1} = \{1,2\}$, $X_{2,2} = \{1,2\} \times \{1,2\}$, and in general,

$$X_{n,m} = \underbrace{\{1,2,\ldots,n\} \times \cdots \times \{1,2,\ldots,n\}}_{m \text{ times}}.$$

Recall that $\mathbf{N} = \{1,2,\ldots,\}$, and let $\mathbf{N}' = \mathbf{N} \cup \{0\} = \{0,1,2,\ldots\}$.

(a) How many elements are there in $X_{n,m}$? Justify your answer.

(b) Let $\text{rem}\colon \mathbf{N} \times \mathbf{N} \to \mathbf{N}'$ be the remainder function. That is, $f(n,m)$ is the remainder when n is divided by m. Prove that rem is surjective by finding at least one element in the preimage $f^{-1}(n)$, for any $n \in \mathbf{N}'$.

(c) Using rem, define a surjective function $f\colon \mathbf{N} \to X_{2,2}$ for which the preimages $f^{-1}(b)$ all have infinitely many elements, for every $b \in X_{2,2}$.

(d) What is the range of $g\colon X_{n,2} \to \mathbf{N}'$, given by $g(b_1,b_2) = \text{rem}(b_1,b_2)$?

(e) What is the range of $h\colon X_{n,3} \to \mathbf{N}'$, given by $h(b_1,b_2,b_3) = \text{rem}(\text{rem}(b_1,b_2),b_3)$?

2. Let $\mathcal{M}$ be the set of all 2×2 matrices filled with real numbers. Define a function $f\colon \mathcal{M} \to \mathbf{R}$ for which $f\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = a \cdot d - b \cdot c$.

(a) Is the function f surjective? Justify your answer.

(b) Is the function f injective? Justify your answer.

(c) Is the function f bijective? Justify your answer.

(d) Prove that $|\mathbf{R}| \leqslant |\mathcal{M}|$ by defining an injection $g\colon \mathbf{R} \to \mathcal{M}$.

(e) Prove or disprove that $|\mathbf{R}| = |\mathcal{M}|$.

Note. The expression $a \cdot d - b \cdot c$ is called the determinant *of M.*

3. You may assume that there exists a bijection $f\colon (0,1) \to [0,1)$.

(a) Find a bijection between $(0,\infty)$ and $(0,1)$.

(b) Find a bijection between $[0,\infty)$ and $[0,1)$.

(c) Find a bijection between $(0,\infty)$ and $[0,\infty)$.

(d) Find a bijection between $(0,\infty)$ and $\mathbf{R}$.

(e) Find a bijection between $\mathbf{R}$ and $\mathbf{R} \times \{0,1\}$.

4. In this question we consider infinite sequences of positive integers, $f(1), f(2), f(3), \ldots$.

(Formally, they are functions from positive integers to positive integers $f : \mathbf{Z}^+ \to \mathbf{Z}^+$, where $\mathbf{Z}^+ = \{1,2,3,\ldots\}$.) Some student wrote various expressions involving quantifiers and the function f.

(a) $\exists a \in \mathbf{Z}^+\ \exists b \in \mathbf{Z}^+\ \forall c \in \mathbf{Z}^+\ (c \geq a\ \rightarrow\ f(c+b) = f(c))$.

(b) $\exists a \in \mathbf{Z}^+ \; \exists b \in \mathbf{Z}^+ \; \forall c \in \mathbf{Z}^+ \; (f(c+b) = f(c) \;\rightarrow\; c \geq a)$.

(c) $\forall a \in \mathbf{Z}^+ \; \exists b \in \mathbf{Z}^+ \; \forall c \in \mathbf{Z}^+ \; (c \geq a \;\rightarrow\; f(c+b) = f(c))$.

(d) $\forall a \in \mathbf{Z}^+ \; \forall c \in \mathbf{Z}^+ \; \exists b \in \mathbf{Z}^+ \; (c \geq a \;\rightarrow\; f(c+b) = f(c))$.

(e) $\exists a \in \mathbf{Z}^+ \; \forall b \in \mathbf{Z}^+ \; \forall c \in \mathbf{Z}^+ \; (b \leq a \rightarrow f(c+b) = f(c))$.

Please check what do these expressions actually say about the properties of the function f. Consider the following sets as possible answers.

- $(\mathbf{Z}^+)^{\mathbf{Z}^+}$ – the set of all infinite sequences of positive integers.
- $\varnothing$ – the *empty set* containing no sequences.
- $\mathcal{A}$ – the set of all *constant sequences.*
- $\mathcal{B}$ – the set of all *eventually constant sequences.*
- $\mathcal{C}$ – the set of all *periodic sequences.*
- $\mathcal{D}$ – the set of all *eventually periodic sequences.*
- $\mathcal{E}$ – the set of all *arithmetic progressions.*
- $\mathcal{F}$ – the set of sequences where, if a number appears, it reappears infinitely often.
- $\mathcal{I}$ – the set of all *injective sequences.*
- $\mathcal{S}$ – the set of all *surjective sequences.*

For every quantifier expression in the list (a)–(e) find the corresponding set ($\mathbf{N}^{\mathbf{N}}$, $\varnothing$, $\mathcal{A}$,...,$\mathcal{S}$). Give short explanations, why do you believe the quantifier expression describes the set of sequences. If it turns out that some expression does not match any of these sets, describe its meaning in plain English.

5. Complete the proofs in Coq notation (they mirror the statements from Homework 2).

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Section traffic.

Inductive City : Type :=
  | A
  | B
  | C.

Variable Plane : City*City -> Prop.
Variable Rail: City*City -> Prop.

Hypothesis PlaneLinks: forall x y:City, x<>y <-> Plane(x,y).
Hypothesis RailLink1: Rail(A,B).
Hypothesis RailLink2: Rail(B,A).
Hypothesis RailLink3: Rail(B,C).
Hypothesis RailLink4: Rail(C,A).
Hypothesis RailLink5: ~Rail(A,A).
Hypothesis RailLink6: ~Rail(A,C).
Hypothesis RailLink7: ~Rail(B,B).
Hypothesis RailLink8: ~Rail(C,B).
Hypothesis RailLink9: ~Rail(C,C).
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(* L1: From any city there is a direct plane-link to some other city. *)
Lemma L1: forall (x: City), exists (y: City), (x<>y /\ Plane(x,y)).
Proof.
  Admitted.

(* It is not true that from any city one can go to any other city
   in two steps like this:
   First take a plane-link and then take a rail-link. *)
Lemma L2: ~(forall (x:City) (y:City),
    exists (z:City), (Plane(x,z) /\ Rail(z,y))).
Proof.
  Admitted.

(* It is generally not true that, if it is possible to go from
   a city 'x' to some other city 'y' with two plane-links,
   then it is also possible to go from 'x' to 'y'
   using a single plane-link. *)
Lemma L3: ~(forall x y z:City, (Plane(x,z) /\ Plane(z,y) -> Plane(x,y))).
Proof.
  Admitted.

End traffic.
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