Homework 7

Discrete Structures
Due Tuesday, February 23, 2021

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1. Let $a_1 = 1$, and for each $n \geqslant 1$, let $a_{n+1} = \sqrt{3 + 2a_n}$. Using induction, prove that for every $n \in \mathbf{N}$, the inequalities $0 \leqslant a_n \leqslant a_{n+1} \leqslant 3$ hold.

   Statement: Let $P(n)$ be the statement "$0 \leqslant a_n \leqslant a_{n+1} \leqslant 3$".

   Base case: Since $a_1 = 1$ and $a_2 = \sqrt{5} \in (2, 3)$, the base case $P(1)$ holds: $0 \leqslant 1 \leqslant \sqrt{5} \leqslant 3$.

   Inductive hypothesis: Suppose that $P(n)$ is true for some $n \geqslant 1$: $0 \leqslant a_n \leqslant a_{n+1} \leqslant 3$.

   Inductive step: From the inductive hypothesis, we have that $0 \leqslant a_{n+1}$.
   Also from the inductive hypothesis we have:

$$\begin{array}{rcrcll} a_n & \leqslant & a_{n+1} & \leqslant & 3 & \text{(from the inductive hypothesis)} \\ 2a_n & \leqslant & 2a_{n+1} & \leqslant & 6 & \text{(multiply by 2)} \\ 3+2a_n & \leqslant & 3+2a_{n+1} & \leqslant & 9 & \text{(add 3)} \\ \sqrt{3+2a_n} & \leqslant & \sqrt{3+2a_{n+1}} & \leqslant & 3 & \text{(take the square root)} \\ a_{n+1} & \leqslant & a_{n+2} & \leqslant & 3 & \text{(definition)} \end{array}$$

   Hence $P(n+1)$ holds.

   Conclusion: By the principle of mathematical induction, $P(n)$ holds for all $n \in \mathbf{N}$. □

2. Let $a_1 = 3$, $a_2 = \frac{3}{2}$, and for each $n \geqslant 3$, let $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$. Using strong induction, prove that for every $n \in \mathbf{N}$, the equality $a_n = 2 + \left(\frac{-1}{2}\right)^{n-1}$ holds.

   Statement: Let $P(n)$ be the statement "$a_n = 2 + \left(\frac{-1}{2}\right)^{n-1}$".

   Base case: When $n = 1$, then $a_1 = 3$ and

$$2 + \left(\frac{-1}{2}\right)^{1-1} = 2 + 1 = 3,$$

   so $P(1)$ holds. When $n = 2$, we have $a_2 = \frac{3}{2}$ and

$$2 + \left(\frac{-1}{2}\right)^{2-1} = 2 - \frac{1}{2} = \frac{3}{2},$$

   so $P(2)$ holds.

   Inductive hypothesis: Suppose that $P(n)$ is true for all $n = 1, \ldots, k$ for some $k \geqslant 2$.

   Inductive step: Since $P(k)$ and $P(k-1)$ hold by the inductive hypothesis, we know that $a_k = 2 + (\frac{-1}{2})^{k-1}$ and $a_{k-1} = 2 + (\frac{-1}{2})^{k-2}$. By definition, we have that

$$\begin{aligned} a_{k+1} &= \frac{1}{2}(a_k + a_{k-1}) \\ &= \frac{1}{2}\left(2 + \left(\frac{-1}{2}\right)^{k-1} + 2 + \left(\frac{-1}{2}\right)^{k-2}\right) \\ &= 1 + \frac{(-1)^{k-1}}{2^k} + 1 + \frac{(-1)^{k-2}}{2^{k-1}} \\ &= 2 + \frac{(-1)^{k-1} + 2\cdot(-1)^{k-2}}{2^k}. \end{aligned} \tag{1}$$

If k is even, then

$$(-1)^{k-1} + 2\cdot(-1)^{k-2} = -1 + 2\cdot 1 = 1,$$

and if k is odd, then

$$(-1)^{k-1} + 2\cdot(-1)^{k-2} = 1 + 2\cdot(-1) = -1.$$

Hence in both cases $(-1)^{k-1} + 2\cdot(-1)^{k-2} = (-1)^k$. The equation from line (1) then becomes

$$a_{k+1} = 2 + \frac{(-1)^{k-1} + 2\cdot(-1)^{k-2}}{2^k} = 2 + \frac{(-1)^k}{2^k} = 2 + \left(\frac{-1}{2}\right)^k,$$

and so $P(k+1)$ holds.

Conclusion: By strong mathematical induction, $P(n)$ holds for all $n \in \mathbf{N}$. □

3. Let $r \in (0,1)$ be an irrational number. Using induction, construct a sequence of nested closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ such that for all $n \in \mathbf{N}$,

   - $r \in I_n$,
   - $I_n \subseteq [0,1]$, and
   - the length of I_n is $\frac{1}{2^n}$.

   The fact that r is irrational would make a difference if I_n were open intervals, but in this case, r could have been rational as well.

   Statement: Let $P(n)$ be the statement "$I_n \subseteq [0,1]$ is a closed interval of length $1/2^n$ with $I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_1$ and $r \in I_n$".

   Base case: Since $r \in (0,1)$, either $r \in [0,\frac{1}{2}]$, in which case let $I_1 = [0,\frac{1}{2}]$, or $r \in [\frac{1}{2},1]$, in which case let $I_1 = [\frac{1}{2},1]$. Then $r \in I_1 \subseteq [0,1]$ and has length $\frac{1}{2}$, so $P(1)$ holds.

   Inductive hypothesis: Suppose that $P(n)$ is true for some $n \geqslant 1$.

   Inductive step: From the inductive hypothesis, we have that $r \in I_n = [a,b]$, with $b - a = \frac{1}{2^n}$. If $r \in [a, a+\frac{1}{2}]$, then let $I_{n+1} = [a, a+\frac{1}{2}]$, else $r \in [b-\frac{1}{2}, b]$, in which case let $I_{n+1} = [b-\frac{1}{2}, b]$. The length of I_{n+1} in either case is

$$\frac{1}{2}(b-a) = \frac{1}{2}\cdot\frac{1}{2^n} = \frac{1}{2^{n+1}}.$$

   Since $I_{n+1} \subseteq I_n \subseteq [0,1]$, $P(n+1)$ holds.

   Conclusion: By the principle of mathematical induction, $P(n)$ holds for all $n \in \mathbf{N}$. □

4. Find the greatest common divisor for the following set of numbers:

$$\left\{7^{n+2} + 8^{2n+1} \,|\, n \in \mathbf{Z}^+\right\}.$$

Prove by induction that the number you found actually divides every element in this infinite set.

Let us prove that the GCD is 57.

Notice that the first two numbers in this set are $855 = 7^3 + 8^3$ and $35169 = 7^4 + 8^5$. Run the Euclidean algorithm:

$$\gcd(35169, 855) = \gcd(855, 114) = \gcd(114, 57) = \gcd(57, 0) = 57.$$

We now prove that every member of the sequence $a_n = 7^{n+2} + 8^{2n+1}$ (where $n \geq 1$) is divisible by 57.

Statement: Let $P(n)$ be the statement: a_n is divisible by 57.

Base case: We verify that $a_1 = 855$ is divisible by 57. Indeed, $855/57 = 15$, which is an integer.

Inductive hypothesis: Assume that for some $k \geq 1$ the statement $P(k)$ is true, i.e. $a_k = 7^{k+2} + 8^{2k+1}$ is divisible by 57.

Inductive step: Now prove the same for $n = k + 1$. Expand the number $a_n = a_{k+1}$:

$$a_{k+1} = 7^{(k+1)+2} + 8^{2(k+1)+1} = 7^{k+2} \cdot 7^1 + 8^{2n+1} \cdot 8^2 =$$

$$= 7 \cdot \left(7^{k+1} + 8^{2k+1}\right) + (8^2 - 7) \cdot 8^{2n+1}.$$

The first term is divisible by 57, since $7^{k+1}+8^{2k+1}$ is divisible by the Inductive Hypothesis. The second term is divisible by 57, since $8^2 - 7 = 57$.

Conclusion: We see that every member of the set is divisible by 57 (and the GCD cannot be larger, since $\gcd(35169, 855) = 57$). □