Homework 7

Discrete Structures

Due Tuesday, February 23, 2021

Submit each question separately in .pdf format

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1. Let $a_1 = 1$, and for each $n \geqslant 1$, let $a_{n+1} = \sqrt{3 + 2a_n}$. Using induction, prove that for every $n \in \mathbf{N}$, the inequalities $0 \leqslant a_n \leqslant a_{n+1} \leqslant 3$ hold.

2. Let $a_1 = 3$, $a_2 = \frac{3}{2}$, and for each $n \geqslant 3$, let $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$. Using strong induction, prove that for every $n \in \mathbf{N}$, the equality $a_n = 2 + \left(\frac{-1}{2}\right)^{n-1}$ holds.

3. Let $r \in (0, 1)$ be an irrational number. Using induction, construct a sequence of nested closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ such that for all $n \in \mathbf{N}$,

   - $r \in I_n$,
   - $I_n \subseteq [0, 1]$, and
   - the length of I_n is $\frac{1}{2^n}$.

4. Find the greatest common divisor for the following set of numbers:

   $$\left\{7^{n+2} + 8^{2n+1} \mid n \in \mathbf{Z}^+\right\}.$$

   Prove by induction that the number you found actually divides every element in this infinite set.