Homework 9

Discrete Structures
Due Tuesday, March 9, 2021

Submit each question separately as .pdf (Except Coq which is `hw7question5.v`)

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1. Let $\mathbf{R}[x]_2 = \{ax^2 + bx + c \ : \ a, b, c \in \mathbf{R}, a \neq 0\}$ be the set of all quadratic polynomials with real coefficients. Let R be the relation between $p, q \in \mathbf{R}[x]_2$, with pRq iff q can be obtained by multiplying p by a real number.

   (a) Is R reflexive? Is R symmetric? Is R antisymmetric? Is R transitive?

   (b) Is R an equivalence relation? If it is, build a subset $S \subseteq \mathbf{R}[x]_2$ such that S contains exactly one representative from each equivalence class.

   Note. In many math texts (also in this course) $\mathbf{R}$ (bold) denotes the set of real numbers. Thus $\mathbf{R}[x]$ is the set of all polynomials with real coefficients, but $\mathbf{R}[x]_2$ are polynomials of degree 2. On the other hand, R (regular font) denotes a relation. Try to distinguish these symbols in your handwritten solutions.

2. Let $S = \{A, B, C\}$.

   (a) Write all the partitions of S.

   (b) Let ρ be the relation on partitions of S, defined by $P_1 \rho P_2$ iff P_2 can be obtained from P_1 by splitting a single class of P_1 into two non-empty subclasses. Represent the relation ρ as a directed graph, labeling the vertices as circles with their partitions written inside.

   (c) Let R be the transitive closure of ρ. How many elements does R contain?

   (d) Is R a partial order? Is it a total order?

3. Let R and S be two relations. Relation R describes friends:

R.ID	**R.Name**	**R.Lastname**
101	Ann	Smith
102	Robert	Jones
103	Jane	Doe

   Relation S describes contact info:

S.ID	**S.Type**	**S.Value**
101	Phone	`123-4567`
101	Email	`ann.smith@rbs.lv`
102	Phone	`456-7890`
104	Email	`maria.brown@rbs.lv`

   (a) Compute the *inner join* of both relations defined by this equality:

$$R \bowtie S = \pi_{R.ID, R.Name, R.Lastname, S.Type, S.Value} \left(\sigma_{R.ID = S.ID} (R \times S) \right). \tag{1}$$

   (b) Compute the following relational algebra expression:

$$(R ⟕ S) = (R \bowtie S) \cup \\ \left((R - \pi_{R.ID, R.Name, R.Lastname} (R \bowtie S)) \times \{(\text{S.Type} : \mathit{null}, \text{S.Value} : \mathit{null})\} \right).$$

(c) Compute the *full outer join* of both relations: Include data from both tables where **R.ID** matches **S.ID**, also include those friends who do not have contact info (and also the contact info without a friend). Fill in the missing fields with null values.

4. Let R be a binary relation on $S = \{v_1, v_2, v_3, v_4\}$ defined by this matrix:

$$M_R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (2)$$

   (a) Show how the Warshall algorithm (Rosen2019, p.637) is run on this matrix to compute the transitive closure R^t. Draw the matrix after every execution of the outer loop (**for** $k := 1$ **to** n). Highlight those values that switch from 0 to 1.
   You will get 4 matrices: $M_R^{(1)}$, $M_R^{(2)}$, $M_R^{(3)}$, $M_R^{(4)}$ – the successive results from the Warshall algorithm as $k = 1, 2, 3, 4$. The last matrix $M_R^{(4)}$ represents the transitive closure R^t.

   (b) Prove or disprove the following statement: "Entry $m_{ij} = 1$ becomes 1 in the k-th iteration of the outer loop ($m_{ij} = 1$ in $M_R^{(k)}$, yet $m_{ij} = 0$ in earlier matrices) iff the shortest path from v_i to v_j contains exactly k steps."
   Is it true for the relation in (2)? Is it true for any binary relation?

5. Complete Homework Exercise 7, Part 2. Its description is given in a separate file, as the initial Coq file `hw7question5.v` .