Homework 10

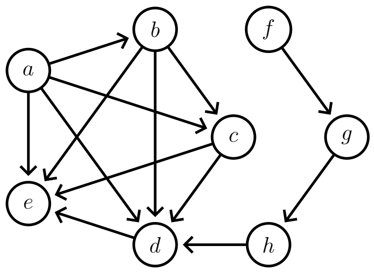
Discrete Structures
Due Tuesday, March 16, 2021

Submit each question separately as .pdf

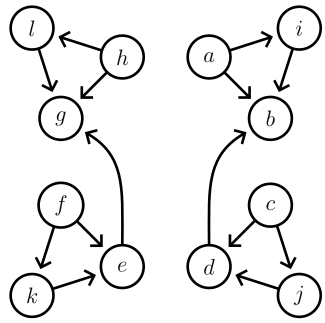
---

1. Let $G_i = (V, E)$ be an directed graph for $i = 1, 2$, and fix $n \in \mathbf{N}$. How many functions $f \colon V \to \{1, \ldots, n\}$ satisfying $f(u) \neq f(v)$ whenever there is a path from u to v are there for each of the following graphs?

$G_1 =$



$G_2 =$



For G_1 and the vertex a, there are n choices, since there are no paths that end at a (we do not consider the empty path starting and ending at a as a path). Similarly:

- for b there are $n-1$ choices (a path starting at a ends at b)
- for c there are $n-2$ choices (paths starting at a, b end at c)
- for f there are n choices (no paths end at f)
- for g there are $n-1$ choices (a path starting at f ends at g)
- for h there are $n-2$ choices (paths starting at f, g end at h)
- for d there are $n-6$ choices (paths starting at a, b, c, f, g, h end at d)
- for e there are $n-7$ choices (paths starting at a, b, c, d, f, g, h end at e)

Hence the total number of functions is:

- 0 if $n = 1, 2, 3, 4, 5, 6, 7$
- $n^2(n-1)^2(n-2)^2(n-6)(n-7)$ if $n \geqslant 8$

For G_2 we follow a similar pattern and find that:

- for a, h, f, c there are n choices
- for i, l, k, j there are $n-1$ choices
- for e, d there are $n-2$ choices
- for g, b there are $n-5$ choices

Hence the total number of functions is:

- 0 if $n = 1, 2, 3, 4, 5$
- $n^4(n-1)^4(n-2)^2(n-5)^2$ if $n \geqslant 6$

$\square$

2. Let $n \in \mathbf{N}$. This questions is about *strings* of length n of the letters `a`, `b`, `c`.

   (a) How many strings contain exactly 10 letters `a`?

   The number of ways to choose 10 spots from n spots is $\binom{n}{10}$. In each of these ways, there are $n-10$ positions left to fill, which can be either the letter `b` or the letter `c`, meaning there are 2^{n-10} possibilities. Hence the total number of string that contain exactly 10 letters `a` is 0 if $n < 10$, and otherwise

$$\binom{n}{10} \cdot 2^{n-10} = \frac{n! \cdot 2^n}{10! \cdot (n-10)! \cdot 2^{10}}.$$

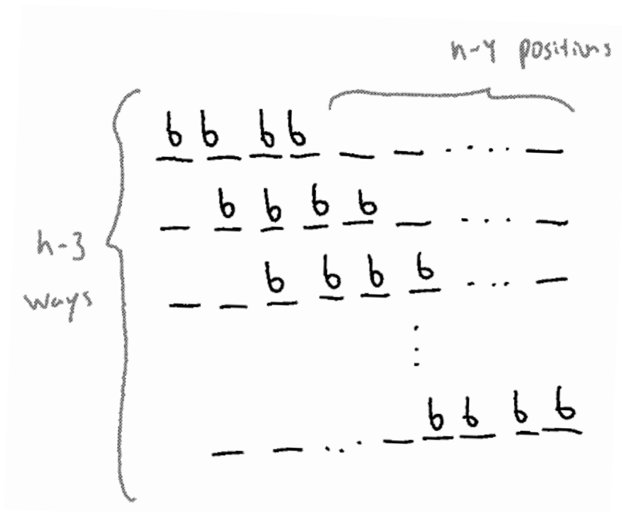
$\square$

   (b) How many strings

   - contain exactly one letter `a`, or
   - contain exactly one substring `bbbb` and no other letters `b`?

   The number of strings that contain exactly one letter `a` is $\binom{n}{1} \cdot 2^{n-1} = n \cdot 2^{n-1}$, going by the answer from part (a) above.

   The number of strings that contain exactly one occurence of the string `bbbb` and no other letters `b` is $(n-3) \cdot 2^{n-4}$, since there are $n-3$ ways to have the string `bbbb`, and in each of these there are 2^{n-4} empty spots left to fill with either `a` or `c`:



   Finally, we must consider how many string have both of these cases. After choosing one of the $n-3$ ways for the string `bbbb`, we must choose one of the remaining $n-4$ positions for the letter `a`. There are $\binom{n-4}{1} = n-4$ such options. Afterwards, the only option for the other $n-5$ positions is the letter `c`, so we have nothing more to choose. Hence the number of ways to have both of these cases is $(n-3)(n-4)$.

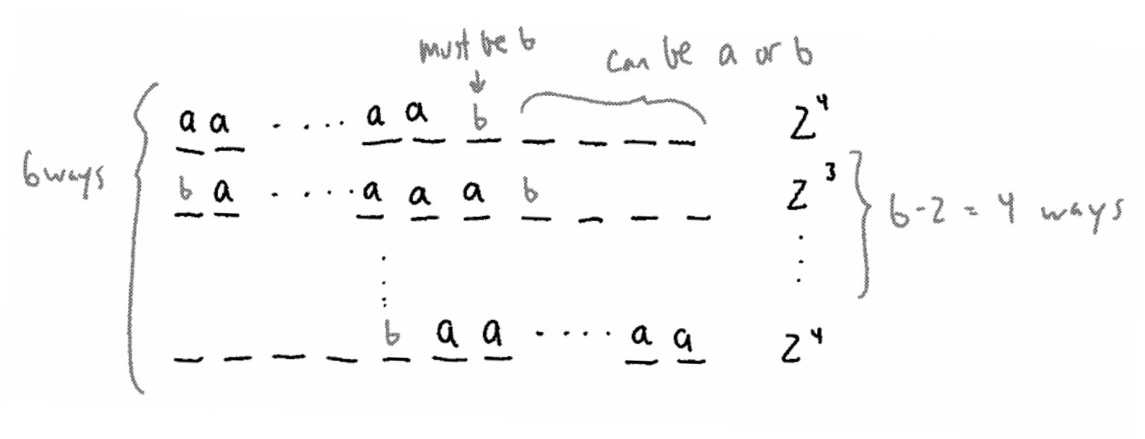
   Putting it all together, the number of strings that contain exactly one letter `a` or one substring `bbbb` and no other letters `b` is 0 for $n < 4$, and otherwise

$$\begin{pmatrix}\text{strings with exactly} \\ \text{one letter } \texttt{a}\end{pmatrix} + \begin{pmatrix}\text{strings with exactly one} \\ \text{substring } \texttt{bbbb} \text{ and no} \\ \text{other letters } \texttt{b}\end{pmatrix} - \begin{pmatrix}\text{strings where both of} \\ \text{these things happen}\end{pmatrix}$$
$$= n \cdot 2^{n-1} + (n-3) \cdot 2^{n-4} - (n-3) \cdot (n-4).$$

$\square$

(c) For $n \geqslant 10$, how many strings contain exactly $n-5$ consecutive letters `a` and contain no letters `c`?

Similarly to part (b) above, there are $6 = n - (n-5) + 1$ ways to have $n-5$ consecutive letters `a` in a strong of length n. There must be no letters `a` before or after this block (else the block will be longer that $n-5$), but otherwise we are free to choose letters `a` or `b`, but not `c`. For 4 of the ways the string of $n-5$ consecutive letters `a` has an empty position before and after it, otheriwse just before or after:



Hence the number of strings that contain exactly $n-5$ consecutive letters `a` and contain no letters `c` is $2 \cdot 2^4 + 4 \cdot 2^3 = 64$. $\square$

3. Let B_n be the set of all compound propositions $f(p_1, \ldots, p_n)$ with n propositional variables. (Compound propositions are considered the same iff they are logically equivalent.)

(a) How many compound propositions f from B_n satisfy this tautology:

$$f(p_1, \ldots, p_n) \rightarrow p_1 \vee \ldots \vee p_n. \tag{1}$$

Answer. There are 2^{2^n-1} such compound propositions (2 to the power $2^n - 1$).
If we drop any requirements about tautologies, n-argument Boolean function has a truth table with 2^n rows (representing all 2^n combinations of `True`/`False` for propositional variables $p_1, p_2, \ldots, p_n$).
Because of tautology (1) we should analyze what truth values make this formula true (for any combination of variables $p_1, \ldots, p_n$). If their disjunction $(p_1 \vee \ldots \vee p_n)$ evaluates to `True`, then we do not need to worry about the value of $f(p_1, \ldots, p_n)$, because the implication will be true no matter what.
On the other hand, if all $p_1, \ldots, p_n$ evaluate to `False` (and also their disjunction is false), then $f(p_1, \ldots, p_n)$ must also be `False` – otherwise the implication (1) becomes false and it is no longer a tautology.
Consequently, we can pick any values for $f(p_1, \ldots, p_n)$ in its truth table with a single exception. We must have

$$f(\underbrace{\texttt{False}, \texttt{False}, \ldots, \texttt{False}}_{\text{All } n \text{ values False}}) = \texttt{False}.$$

Therefore we can freely choose only $2^n - 1$ values in the truth table, so the count of such functions is 2^{2^n-1}. □

(b) How many compound propositions f from B_n satisfy this tautology:

$$f(p_1, \ldots, p_n) \to p_1 \oplus \ldots \oplus p_n. \tag{2}$$

Answer. There are $2^{2^{n-1}}$ such compound propositions (2 to the power 2^{n-1})
Here the answer is obtained in a similar way as in the previous sample. But unlike the earlier tautology, in (2) there is exactly half of all 2^n rows in the truth table where the subexpression $p_1 \oplus \ldots \oplus p_n$ turns `False`, so the Boolean function $f(p_1, \ldots, p_n)$ must be `False` as well.

$$f(\underbrace{\texttt{True}, \texttt{True}, \texttt{False}, \ldots, \texttt{False}}_{\text{Any even number of } n \text{ values True}}) = \texttt{False}.$$

Therefore, only 2^{n-1} truth values can be picked in any way we like. Assigning arbitrary True/False value in 2^{n-1} slots gives us $2^{2^{n-1}}$ possibilities. □

4. Someone selected k points on the plane: $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, ..., $A_k(x_k, y_k)$. All of them have both integer coordinates, no three points are on the same line.

(a) How many triangles can be created from these points?

$\binom{k}{3} = \frac{k!}{(k-3)!3!}$ describes in how many ways one can pick unordered set of 3 vertices out of k vertices. And every set of 3 vertices makes a new triangle, since no three points are on the same line. □

(b) What is the smallest value k for which at least one of the line segments A_iA_j will have its midpoint with both integer coordinates?

Answer. For $k = 5$ points at least one midpoint of some segment A_iA_j will have both integer coordinates.
First we check that for $k = 4$ the statement is not true. If we pick $A_1(0; 0)$, $A_2(1; 0)$, $A_3(1; 1)$, $A_4(0; 1)$ to be the vertices of a unit square, then all the midpoints of segments A_iA_j will have at least one non-integer coordinate.
If $k = 5$, then consider the following 4 categories (S_1, S_2, S_3, S_4) of points with integer coordinates:

$$\begin{cases} S_1 := \{(x_i, y_i) \mid (x_i \equiv 0 \pmod 2) \wedge (y_i \equiv 0 \pmod 2)\} \\ S_2 := \{(x_i, y_i) \mid (x_i \equiv 0 \pmod 2) \wedge (y_i \equiv 1 \pmod 2)\} \\ S_3 := \{(x_i, y_i) \mid (x_i \equiv 1 \pmod 2) \wedge (y_i \equiv 0 \pmod 2)\} \\ S_4 := \{(x_i, y_i) \mid (x_i \equiv 1 \pmod 2) \wedge (y_i \equiv 1 \pmod 2)\} \end{cases}$$

Intuitively, any point is either (even,even), or (even, odd), or (odd, even), or (odd,odd). As soon as you have $k = 5$ points at least two will be in the same category (S_1, S_2, S_3, or S_4); this happens because of the Pigeonhole principle. □

(c) What is the smallest value k for which at least one of the triangles will have its *centroid* (the point where all its medians meet) in a point with both integer coordinates?

Answer. For $k = 9$ points at least one triangle $A_iA_jA_k$ will have its centroid in a point with both integer coordinates.

First, notice that a triangle ABC, where points have coordinates $A(x_A, y_A)$, $B(x_B, y_B)$ and $C(x_C, y_C)$ has the centroid (intersection of its medians in the following point:

$$\left(\frac{x_A + x_B + x_C}{3}, \frac{y_A + y_B + y_C}{3}.\right) \tag{3}$$

The formula to find centroid coordinates is easy to find with a Web search (`https://bit.ly/3eVlhcl`), one can also derive it independently (by knowing that the centroid splits all medians in ratio $2:1$).

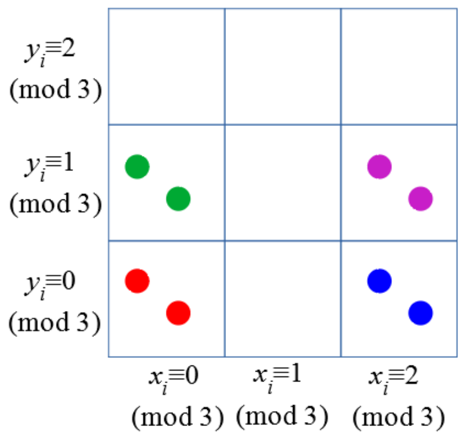


Figure 1: Classifying points into nine categories.

For this reason, having **both** coordinate sums simultaneously divisible by 3 is the only relevant condition. We classiify all plane points with integer coordinates into 9 categories (depending on their remainders when divided by 3). For example, any of the 8 points in Figure 2 (after we compute the remainders for their x and y coordinates) maps to some little circle in Figure 1.

Since we have to find the **least** number of points that guarantee a centroid with all integer coordinates, our proof will consist of two parts:

(1) Counterexample for $k = 8$. We show that it is possible to select $k = 8$ points so that none of the $C_8^3 = 56$ triangles that can be made out of these points has its centroid with both integer coordinates.

(2) General proof for $k = 9$. We show that no matter how you locate $k = 9$ points, there will be a triangle having centroid in integer points.

(1) Counterexample for $k = 8$. Consider figure 2. In that image 8 points are selected, no three points are on the same line. As shown in 1, their coordinates have remainders 0 or 1 when dividied by 3 (there are two points in each of the four possible categories). No matter how we pick 3 points out of these 8, they cannot be all from the same category. Either x or y coordinates will not be all congruent modulo 3. So, their sum can be either $0 + 0 + 1$ or $0 + 1 + 1$ – in either case it is not divisible by 3.

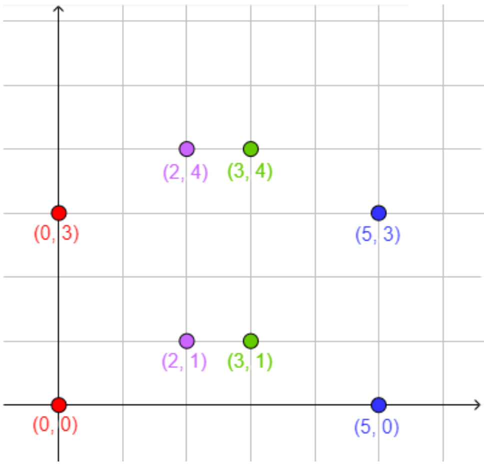


Figure 2: Plane with $k = 8$ points selected.

(2) General proof for $k = 9$. We reason by contradiction. Imagine that we have somehow selected 9 points so that there is no triangle with an integer centroid. Then all 9 points can be placed on a 3×3 grid. There are some combinations that we must avoid, because they would immediately lead to an integer centroid; see Figure 3. They include four cases:
Condition (A) Three points in the same category,
Condition (B) Three points in different slots of the same row,
Condition (C) Three points in different slots of the same column,
Condition (D) Three points on pairwise different rows and columns.

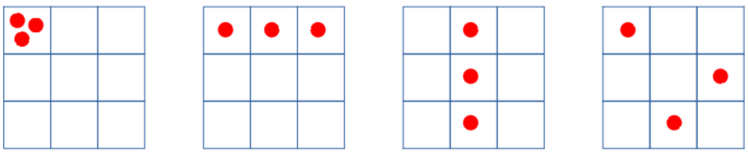


Figure 3: Configurations that create integer centroids.

Since we have 9 points (and because of Condition (A)) no three points can be in the same category, by the Pigeonhole principle at least 5 categories should contain some points (since $\lfloor 9/4 \rfloor = 3$; using just 4 categories would lead to 3 points ending up in the same category).
Because of Condition (B) we cannot have three non-empty categories on the same row. For that reason, the five non-empty categories should be distributed in rows like this: $2 + 2 + 1$ (or, perhaps $2 + 1 + 2$ or $1 + 2 + 2$). Let us assume that the top two rows contain two non-empty categories each (other cases are similar). See Figure 4 – a cell is shaded, if there is at least one point in this category. We have two possibilities: the first two rows could be filled in identically (picture to the left), or, perhaps, they can partially overlap (picture to the right). In either case we cannot find another, the 5th category to fill in, because it will violate either the Condition (C) or the Condition (D) (see Figure 3).
This is a contradiction, therefore it is not possible to have 9 points $A_1, \ldots, A_9$ such that there is no triangle with its centroid with both integer coordinates. Since we had a counterexample for $k = 8$, the value $k = 9$ is indeed the smallest value. $\square$

5. Because of epidemiological safety measures only one robot is allowed to visit the public

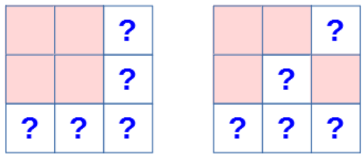


Figure 4: How to fill in the first two rows (two non-empty categories each).

library. Every day the robot arrives to a shelf with 10 volumes of an encyclopedia and reorders them so that volume from the slot #1 goes to the slot n_1, the volume from #2 goes to the slot n_2, and so on. ($n_1, \ldots, n_{10}$ are different integers between 1 and 10; they are the same every day.) The robot observes that after T days the volumes return to the original order.

(a) What is the value of T, if we define $n_k = (6 \cdot k \bmod 11)$ for $k = 1, \ldots, 10$?

We have $T = 10$. Indeed, consider one book (say, book #1). Every time it travels from the location k to the location $(6 \cdot k \bmod 11)$, i.e. it moves to the location that is congruent to $6 \cdot k \pmod{11}$. We are now ready to write the trajectory for the book #1:

$$1 \to 6 \to 3 \to 7 \to 9 \to 10 \to 5 \to 8 \to 4 \to 2 \to 1.$$

The book visited all 10 places before returning back to its original location.

Because of the Little Fermat theorem any other book will return to its original value as well, since $6^{10} \equiv 1 \pmod{11}$. □

(b) Somebody modified robot's software in such a way that $T > 10$ (the books need more than 10 days to return to their initial state). Provide some example of the values n_k when this happens and find the corresponding T.

We can have $T = 30$. Let us consider how to construct it. We divide the books into three groups: $\{1, 2, 3, 4, 5\}$, $\{6, 7, 8\}$, $\{9, 10\}$. In every group the robot reorders books in a cycle. Here is the permutation that ensures these cycles:

$$\left(\begin{array}{ccccc|ccc|cc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 & 10 & 9 \end{array} \right).$$

After every 5 steps the book #1 will return to its original position.
After every 3 steps the book #6 will return to its original position.
After every 2 steps the book #9 will return to its original position.

So, if the step number is divisible by 30, all the three cycles will return to their original positions. It will not return earlier, because no number $T < 30$ is divisible by all 5, 3, and 2.

By picking different group sizes one can get other periods longer than 10. For example, $3 \cdot 4 \cdot 1 \cdot 1 \cdot 1 = 12$, $3 \cdot 5 \cdot 1 \cdot 1 = 15$, $4 \cdot 5 \cdot 1 = 20$, or $3 \cdot 7 = 21$. □