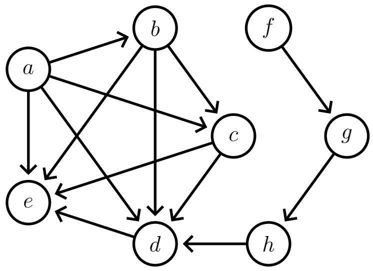
Homework 10

Discrete Structures
Due Tuesday, March 16, 2021

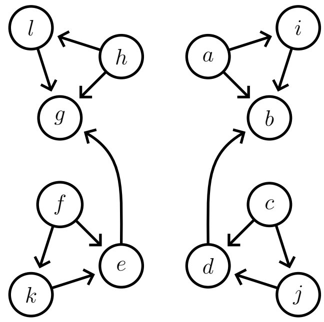
Submit each question separately as .pdf

1. Let $G_i = (V, E)$ be an directed graph for $i = 1, 2$, and fix $n \in \mathbf{N}$. How many functions $f: V \to \{1, \ldots, n\}$ satisfying $f(u) \neq f(v)$ whenever there is a path from u to v are there for each of the following graphs?

$G_1 =$



$G_2 =$



2. Let $n \in \mathbf{N}$. This questions is about *strings* of length n of the letters `a`, `b`, `c`.
   (a) How many strings contain exactly 10 letters `a`?
   (b) How many strings
      - contain exactly one letter `a`, or
      - contain exactly one substring `bbbb` and no other letters `b`?

   (c) For $n \geqslant 10$, how many strings contain exactly $n-5$ consecutive letters `a` and contain no letters `c`?

3. Let B_n be the set of all compound propositions $f(p_1, \ldots, p_n)$ with n propositional variables. (Compound propositions are considered the same iff they are logically equivalent.)
   (a) How many compound propositions f from B_n satisfy this tautology:
   $$f(p_1, \ldots, p_n) \to p_1 \vee \ldots \vee p_n. \qquad (1)$$
   (b) How many compound propositions f from B_n satisfy this tautology:
   $$f(p_1, \ldots, p_n) \to p_1 \oplus \ldots \oplus p_n. \qquad (2)$$

4. Someone selected k points on the plane: $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, ..., $A_k(x_k, y_k)$. All of them have both integer coordinates, no three points are on the same line.
   (a) How many triangles can be created from these points?
   (b) What is the smallest value k for which at least one of the line segments A_iA_j will have its midpoint with both integer coordinates?
   (c) What is the smallest value k for which at least one of the triangles will have its *centroid* (the point where all its medians meet) in a point with both integer coordinates?

5. Because of epidemiological safety measures only one robot is allowed to visit the public library. Every day the robot arrives to a shelf with 10 volumes of an encyclopedia and reorders them so that volume from the slot #1 goes to the slot n_1, the volume from #2 goes to the slot n_2, and so on. ($n_1, \ldots, n_{10}$ are different integers between 1 and 10; they are the same every day.) The robot observes that after T days the volumes return to the original order.

   (a) What is the value of T, if we define $n_k = (6 \cdot k \bmod 11)$ for $k = 1, \ldots, 10$?

   (b) Somebody modified robot's software in such a way that $T > 10$ (the books need more than 10 days to return to their initial state). Provide some example of the values n_k when this happens and find the corresponding T.