Homework 11

Discrete Structures
Due Tuesday, March 23, 2021

Submit each question separately as .pdf

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1. Suppose you have two dice, one with 6 sides (having the numbers $1, \ldots, 6$) and one with 8 sides (having the numbers $1, \ldots, 8$). You roll both at the same time. Let T be the sum total of numbers on both die that are rolled.

   (a) Express all the possible values of T and the number of ways each value could be rolled.

   (b) Let T_{avg} be the arithmetic mean of all the possible values of T from part (a). What is the probability of rolling exactly T_{avg}?

   (c) What is the probability of rolling exactly $T_{avg} - 1$ or exactly $T_{avg} + 1$?

2. A bank assigns 5 digit PIN's (for example, 02270) to bank cards (customers don't get to pick their own PIN's.) Assume all combinations of 5 digits are equally likely. You find a bank card belonging to this bank left in a bank machine.

   (a) What is the probability of guessing the PIN if you try three times using a different possible PIN each time?

   (b) What is the probability the PIN has 5 different digits?

   (c) What is the probability the PIN contains at least one repeated digit?

   (d) If a bank card is stolen do you think a PIN with no repeated digits is more safe or less safe than one with repeated digits? Why?

3. Assume that we roll two regular dice (they can roll numbers 1–6 with equal probabilities). Define the following events:

$$\begin{cases} A := \text{the sum of the points is 7,} \\ B := \text{the first die rolled a 2,} \\ C := \text{the second die rolled a 5.} \\ D := \text{the sum of the points is at least 7,} \end{cases}$$

   (a) What are the conditional probabilities $p(B|A)$ and $p(B|D)$?

   (b) Are A, B, C pairwise independent?

   (c) Are A, B, C mutually independent?

4. A chip factory *Intel* adds one toy animal to every bag of chips. There are three sorts of animals – Aligators, Bears or Cats (each one appears with probability p = 1/3). Let the random variable X denote the chip bags that someone needs to purchase in order to collect 3 different animals. Find the expected value $E(X)$.
   Hint. The total number of the bags to open to collect all three toy animals is denoted by random variable X. (It can take values $3, 4, 5, \ldots$ with certain probabilities). It may be tricky to compute these probabilities directly. Instead, you can express $X = X_1 + X_2 + X_3$ a sum of three simpler random variables:
   X_1 shows the bags that were needed to get the 1st unique animal (whatever it is),
   X_2 shows the bags needed to get the 2nd unique animal (already having the 1st one),

X_3 shows the bags needed to get the 3rd unique animal (after getting the first two). All the random variables (except X_1) follow the geometric distribution (Textbook p.510, chapter 7.4.5).

5. There are n people in some city after its reopening. Assume that health officials know that the fraction p of them still have a viral disease (here $p \in (0; 1)$ is a positive number very close to 0). There is a test for this disease which is 100% accurate; no false positives or false negatives. In order to save the number of tests needed, the people were grouped into groups of size k. (Assume that n is large and divisible by k.)

   For each group, the health officials blended k samples taken from the people in the group, and tested the blended sample. If everyone in the group is healthy, they need just one test. If the group tests positive for the disease, then they need to test all k people in the group, so they spend $k+1$ tests for that group. Let X be a random variable denoting the total number of tests spent to test all n people in the city in this manner.

   (a) Express the expected value $E(X)$ in terms of n, the fraction of infected people p and the group size k?

   (b) What group size k makes the value $E(X)$ as small as possible? (You can assume that p is so small that $(1-p)^k \approx 1$ and $\ln(1-p) \approx -p$.)