Homework 12

Discrete Structures
Due Tuesday, March 30, 2021

Submit each question separately as .pdf

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1. This question is about strings of the letters `a` and `b`. A "valid" string $\ell_1 \cdots \ell_n$ of length n is a string for which $\ell_1 \cdots \ell_k$ contains at least as many letters `a` as letters `b`, for every $k = 1, \ldots, n$.

   (a) Give all the valid strings of length 1,2,3,4.

   The strings are:

   $$\begin{array}{cccc} a & aa & aaa & aaaa \\ & ab & aab & aaab \\ & & aba & aaba \\ & & & aabb \\ & & & abaa \end{array}$$

   □

   (b) You randomly choose a string of the letters `a` and `b`, of length between 1 and 4 inclusive, and the string is valid. What is the probability that the string you chose has length 3?

   Bayes theorem □

   (c) Find the recurrence relation for valid strings of length n.
   Hint: Split up your relation into cases when n is even or odd.

2. Consider the recurrence relation $a_n = 3ea_{n-1} - 2e^2 a_{n-2} - F(n)$, with

   $$F(n) = (e-1)(e-2)2^{n-2}, \qquad a_0 = \pi, \qquad a_1 = \frac{3\pi}{2}.$$

   (a) What is the associated homogeneous recurrence relation and what are the roots of its characteristic equation?

   (b) Find a solution to the associated homogeneous recurrence relation.

   (c) Find a particular solution to the recurrence relation.

   Using Theorem 6 from page 549, a particular solution is $a_n = p_0 2^n$. Placing this into the recurrence relation, we get

   $$\begin{aligned} p_0 2^n &= 3ep_0 2^{n-1} - 2e^2 p_0 2^{n-2} - (e-1)(e-2)2^{n-2} \\ p_0(2^n - 3e2^{n-1} + 2e^2 2^{n-2}) &= -(e-1)(e-2)2^{n-2} \\ p_0 2^{n-1}(2 - 3e + e^2) &= -(e-1)(e-2)2^{n-2} \\ p_0 2^{n-1}(2-e)(1-e) &= -(e-1)(e-2)2^{n-2} \\ p_0 &= -\frac{2^{n-2}}{2^{n-1}} \\ p_0 &= -\frac{1}{2}. \end{aligned}$$

   Hence a particular solution is $a_n = -2^{n-1}$. □

(d) Find the general solution to the recurrence relation.

3. In the game of Hanoi towers the goal is to move n different disks from Peg 1 to Peg 3 (using also Peg 2 when necessary) moving one disk at a time and never placing a larger disk on top of a smaller disk. Assume that the disks have costs associated with moving (moving the smallest disk once costs 1 unit, moves of the next disks cost $2, 3, \ldots, n$ units respectively). Let G_n be the total cost to move all disks from Peg 1 to Peg 3.

   (a) Define G_n as a recurrent sequence.

$$G_1 = 1; \quad G_n = 2G_{n-1} + n.$$

To verify these formulas, note that moving just one small disk costs 1 unit ($G_1 = 1$). Furthermore, moving n disks (weights $1, \ldots, n$) from Peg 1 to Peg 3 can be expressed by these three actions:

- Move the smaller set of disks (weights $1, \ldots, n-1$) from Peg 1 to Peg 2. (Cost is G_{n-1}.)
- Move the largest disk (weight n) from Peg 1 to Peg 2. (Cost is n.)
- Move the smaller set of disks (weights $1, \ldots, n-1$) from Peg 2 to Peg 3. (Cost is once again G_{n-1}.)

Therefore $G_n = G_{n-1} + n + G_{n-1} = 2G_{n-1} + n$. □

   (b) Find a closed formula for this sequence.

We prove by induction that $G_n = 2^{n+1} - (n+2)$.
Base Case: $n = 1$.
By recursive formula $G_1 = 1$. And the closed expression is $2^2 - 3 = 4 - 3 = 1$.
Inductive Step: $n = k$ *implies* $n = k+1$.
Inductive hypothesis: Assume $G_k = 2^{k+1} - (k+2)$.
Now we should prove that $G_{k+1} = 2^{k+2} - (k+3)$.
Rewrite G_{k+1} using recurrent formula and substitute the inductive hypothesis:

$$G_{k+1} = 2G_k + (k+1) = 2\left(2^{k+1} - (k+2)\right) + (k+1) = 2 \cdot 2^{k+1} - 2 \cdot (k+2) + (k+1).$$

The latter expression can be simplified as $2^{k+2} - (2k+4) + (k+1) = 2^{k+2} - (k+3)$. This is exactly the equality we had to prove. □

4. Define a recurrent sequence $f(n) = 3f(n/3) + 3n$, $f(1) = 1$.

   (a) Use Master theorem to find a function $g(n)$ such that $f(n)$ is in $O(g(n))$.

Master theorem (Rosen2019, p.558) can solve all recurrences in the following form:

$$f(n) = af(n/b) + cn^d,$$

where the initial conditions (the value of $f(1)$) can be arbitrary, and a, b, c, d are numbers (in particular, $a \geq 1$, $b \geq 2$, $c > 0$, $d \geq 0$; also b is an integer, since divide-and-conquer recurrences subdivide the problem into an integer number of parts).

In our case, $a = 3$, $b = 3$, $c = 3$, $d = 1$.
Therefore, we use the 2nd case in the Master theorem (when $a = b^d$), and conclude that $f(n)$ is in $O(n^d \log n)$ or by replacing $d = 1$ we get $O(n \log n)$.
So the function $g(n) = n \log n$.
This recurrence is very similar to the Merge Sort. In fact, we could get exactly this recurrence when estimating the time complexity for algorithms that are doing a variant of Merge Sort (subdividing the array in 3 equal parts every time). □

(b) Find $f(3^{10})$.

We can rewrite the recurrence 10 times:

$$\begin{aligned} f(3^{10}) &= 3f(3^9) + 3 \cdot 3^{10} = \\ &= 3(3f(3^8) + 3 \cdot 3^9) + 3 \cdot 3^{10} = \\ &= 3^2 f(3^8) + 3^{11} + 3^{11} = \\ &= 3^2(3f(3^7) + 3 \cdot 3^8) + 3^{11} + 3^{11} = \\ &= 3^3 f(3^7) + 3^{11} + 3^{11} + 3^{11} = \\ &= \ldots = \\ &= 3^{10} f(3^0) + \underbrace{3^{11} + 3^{11} + \ldots + 3^{11}}_{10 \text{ times}} = \\ &= 3^{10} \cdot 1 + 10 \cdot 3^{11} = (1 + 3 \cdot 10) \cdot 3^{10} = 31 \cdot 3^{10} = \\ &= 31 \cdot 59049 = 1830519. \end{aligned}$$

One can prove by induction that $f(3^k) = (1 + 3k)3^k$. So the functions in Big-O-Notation class $O(n \log n)$ can have integer values, if the function argument has special form such as 3^k. □

5. There are two identical decks of $2N$ playing cards. Each deck is shuffled and laid on the table in a single line. Event $E_{3;2N}$ means that there are exactly three matches between the two lines of cards.

(a) Prove that the probability $p(E_{3;2N})$ is expressed by the formula:

$$p(E_{3;2N}) = \frac{1}{3!}\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{1}{(2N-4)!} - \frac{1}{(2N-3)!}\right) = \frac{1}{3!}\sum_{k=0}^{2N-3}(-1)^k \frac{1}{k!}.$$

If we have two decks of $2N$ cards each, then formally we could build $((2N)!)^2$ sequences of these cards – each deck can be laid in $(2N)!$ different ways. Let us assume that we only care about the pairs (card on the top line vs. the card on the bottom line), but the order of the pairs does not matter. For the sake of simplicity we assume that the top line is ordered in some order.
See Figure 1 on how the pairs can be laid out in a "canonical order" preserving all matches. After reordering we see that there are just $(2N)!$ ways to create pairs of playing cards between two decks. Let us count how many of these $(2N)!$ ways create exactly 3 matches.

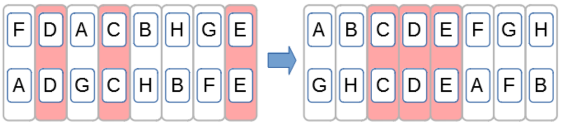

Figure 1: Matches preserved after reordering pairs.

First, note that there are $\binom{2N}{3}$ ways to pick the three matching cards. All the other $(2N-3)$ cards make a derangement, i.e. permutation where no element stays in its previous location (as it would increase the total number of matches, but we need exactly three matches).
By Theorem 2 (Rosen2019, p.589) the number of derangements can be expressed by the following formula:

$$D_{2N-3} = (2N-3)!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{2N-3}\frac{1}{(2N-3)!}\right).$$

The last term has minus sign, since $(-1)^{2N-3} = -1$. We can now multiply the number of matches (3 out of $2N$) with the number of derangements on the remaining $2N-3$ elements:

$$\#\text{Events}_{\text{with 3 matches}} = \binom{2N}{3} \cdot (2N-3)!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots - \frac{1}{(2N-3)!}\right) = \\ = \frac{(2N)!}{(2N-3)!3!} \cdot (2N-3)!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots - \frac{1}{(2N-3)!}\right) = \\ \frac{(2N)!}{3!} \cdot \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots - \frac{1}{(2N-3)!}\right).$$

Since all $(2N)!$ permutations are equally likely, divide this expression by $(2N)!$ to get the probability:

$$p(E_{3;2N}) = \frac{\#\text{Events}_{\text{with 3 matches}}}{(2N)!} = \frac{1}{3!}\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{1}{(2N-4)!} - \frac{1}{(2N-3)!}\right).$$

This is exactly the formula we had to prove. □

(b) Find the limit: $\lim_{N\to\infty} p(E_{3;2N})$.

(A *match* means the same card in the same position. For example, if $N = 4$, then the following two lines of 8 cards match for these three cards: D, C, E.)

```
F D A C B H G E
A D G C H B F E
```

We will prove that

$$\lim_{N\to\infty} p(E_{3;2N}) = \frac{1}{6}e^{-1} \approx 0.06131324.$$

Namely, the chance to have exactly 3 matches is about 6.13% if the number of cards $2N$ is large (in the limit this probability does not depend on N very much; it converges fast to this value).

First note that Taylor series to compute function $f(x) = e^x$ is the following:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

This series converges for every $x \in \mathbf{R}$. By substituting the value $x = -1$ we get the following:

$$e^{-1} = 0.367879441171442\ldots = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots.$$

The expression proven in **(a)** is the partial sum of this series (the first $2N - 2$ members are only written), and it is divided by 3!. Therefore the limit is $\frac{1}{6}e^{-1}$. $\square$