Homework 12

Discrete Structures
Due Tuesday, March 30, 2021

Submit each question separately as .pdf

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1. This question is about strings of the letters `a` and `b`. A "valid" string $\ell_1 \cdots \ell_n$ of length n is a string for which $\ell_1 \cdots \ell_k$ contains at least as many letters `a` as letters `b`, for every $k = 1, \ldots, n$.

   (a) Give all the valid strings of length 1,2,3,4.

   (b) You randomly choose a string of the letters `a` and `b`, of length between 1 and 4 inclusive, and the string is valid. What is the probability that the string you chose has length 3?

   (c) Find the recurrence relation for valid strings of length n.
   Hint: Split up your relation into cases when n is even or odd.

2. Consider the recurrence relation $a_n = 3ea_{n-1} - 2e^2 a_{n-2} - F(n)$, with

$$F(n) = (e-1)(e-2)2^{n-2}, \qquad a_0 = \pi, \qquad a_1 = \frac{3\pi}{2}.$$

   (a) What is the associated homogeneous recurrence relation and what are the roots of its characteristic equation?

   (b) Find a solution to the associated homogeneous recurrence relation.

   (c) Find a particular solution to the recurrence relation.

   (d) Find the general solution to the recurrence relation.

3. In the game of Hanoi towers the goal is to move n different disks from Peg 1 to Peg 3 (using also Peg 2 when necessary) moving one disk at a time and never placing a larger disk on top of a smaller disk. Assume that the disks have costs associated with moving (moving the smallest disk once costs 1 unit, moves of the next disks cost $2, 3, \ldots, n$ units respectively). Let G_n be the total cost to move all disks from Peg 1 to Peg 3.

   (a) Define G_n as a recurrent sequence.

   (b) Find a closed formula for this sequence.

4. Define a recurrent sequence $f(n) = 3f(n/3) + 3n$, $f(1) = 1$.

   (a) Use Master theorem to find a function $g(n)$ such that $f(n)$ is in $O(g(n))$.

   (b) Find $f(3^{10})$.

5. There are two identical decks of $2N$ playing cards. Each deck is shuffled and laid on the table in a single line. Event $E_{3;2N}$ means that there are exactly three matches between the two lines of cards.

   (a) Prove that the probability $p(E_{3;2N})$ is expressed by the formula:

$$p(E_{3;2N}) = \frac{1}{3!}\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{1}{(2N-4)!} - \frac{1}{(2N-3)!}\right) = \frac{1}{3!}\sum_{k=0}^{2N-3}(-1)^k \frac{1}{k!}.$$

(b) Find the limit: $\lim_{N\to\infty} p(E_{3;2N})$.

(A *match* means the same card in the same position. For example, if $N = 4$, then the following two lines of 8 cards match for these three cards: `D`, `C`, `E`.)

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F D A C B H G E
A D G C H B F E
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