Homework 13

Discrete Structures
Due Friday, April 16, 2021

Submit each question separately as .pdf

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1. Recall that an undirected graph is *k-regular* if every vertex has degree k. Prove that a $2k$-regular graph has no cut edges, for every $k \in \mathbf{N}$.

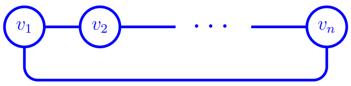
   This will be a proof by contradiction, so we assume there exists a cut edge e. Without loss of generality, assume that $G = (V, E)$ is connected (if G is not connected, choose the connected component containing e, and call that G). After removing e, there are two connected components $G_a = (V_a, E_a)$ and $G_b = (V_b, E_b)$, with $a \in V_a$ and $b \in V_b$. Every vertex in G_a and G_b has degree $2k$ except a and b, which have degree $2k - 1$. By the handshaking theorem for G_a, we have

   $$2|E_a| = \sum_{v \in V_a} \deg(v) = 2k(|V_a| - 1) + 2k - 1,$$

   and similarly for G_b. However, the number on the left is even, but the number on the right is odd, which is a contradicition. □

2. Let $G = (V, E)$ be bipartite. Prove that G does not have C_n as a subgraph, for n odd.

   This will be a proof by contradiction, so we assume there exists a subgraph C_n of G, for n odd. Let $v_1, \ldots, v_n$ be the vertices of the cycle in order, as below.

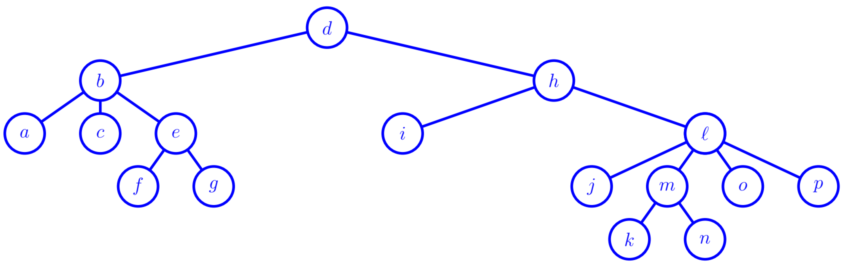
   Since G is bipartite, the vertex set V is decomposed as a union $V_1 \cup V_2$ of disjoint sets (that is, $V_1 \cap V_2 = \emptyset$). Without loss of generality, suppose that $v_1 \in V_1$. This means that $v_2 \in V_2$, which then implies that $v_3 \in V_1$ as well. Continuing this, we get that $v_1, v_3, v_5, \ldots, v_n \in V_1$, since n is odd. However, there is an edge $\{v_n, v_1\}$, and both $v_1, v_n \in V_1$ are in the same partition. This is a contradiction, as there cannot be edges between vertices of the same partition. Hence G cannot have C_n as a subgraph. □

3. Construct an ordered rooted tree whose postorder traversal is

   $$a, c, f, g, e, b, i, j, k, n, m, o, p, \ell, h, d.$$

   In this graph the vertex ℓ has four children, b has three children, d, e, h, m have two children each, and all other vertices are leaves.

   The tree is given below.

   □

4. Let G be a graph with 100 vertices with the following property: The graph G does not contain K_3 as a subgraph. Estimate the largest possible number of edges in G.
Note. An estimate has 2 parts. A *lower bound* shows a graph $G = (V, E)$ with the property and a possibly large number of edges $|E| = m_1$. An *upper bound* proves that for $|E| = m_2$ the property must fail. (Ideally, $m_2 = m_1 + 1$; it would be the exact estimate.)

Claim 1 (Lower bound). There exists a graph not containing triangles with $2n$ vertices and n^2 edges.

Proof. This can be reached in a complete bipartite graph $K_{n,n}$. It contains n vertices in one partition; another n vertices in another partition; and all pairs of vertices in opposite partitions are connected. This graph does not contain any triangles (as triangle is not a bipartite graph). And also it has $n \cdot n = n^2$ edges. ■

Claim 2 (Upper bound). Any graph with $2n$ vertices and not containing a triangle K_3 as a subgraph has n^2 vertices or less.

Proof. We prove this by induction. Note that for $n = 1$ we have a 2-vertex graph; it can have one edge connecting both vertices, i.e. $n^2 = 1^2 = 1$.
Inductive hypothesis $n = k$. Assume that any graph with $2n$ and without triangles there are actually up to k^2 edges.

We now set the vertex count $n = k + 1$. We must prove that there are no more than $(k+1)^2$ edges in such graph. First, note that an optimal (largest number of edges) graph G contains at least one edge (empty graph would not be optimal). Let (u, v) be an edge in the optimal graph G with $2(k + 1)$ vertices. We claim that there canot be two edges (u, w) and (v, w) (for any w), because then u, v, w would be a triangle. Therefore either u or v can connect to any of the $2k$ remaining vertices.

By assumption, G (minus two vertices u and v) is a $2k$-vertex graph. If we add 1 (the edge (u, v) itself) and then also $2k$ (the number of vertices that either u or v (but never both!) have visited). Therefore the number of vertices in graph G is

$$|E| \leq k^2 + 1 + 2k = (k + 1)^2.$$

This completes the proof by induction. ■ □

5. A computer game uses a labyrinth – the directed graph shown in Figure 1. In the beginning a ghost enters one of the 5 rooms A, B, C, D or E (any room with the same probability $p = 0.2$). During the first step the ghost randomly chooses one of the outbound edges of its current room and moves to another room; during the next step it takes another outbound edge from its current state and so on.

   (a) Find the probabilities for every room where the ghost will be after one, two and three steps.

   Initially the probabilities are represented by a vector

   $$\left(p_A^{(0)}, p_B^{(0)}, p_C^{(0)}, p_D^{(0)}, p_E^{(0)}\right) = (0.2, 0.2, 0.2, 0.2, 0.2).$$

   Assume that the ghost has made one step. For every vertex $v \in \{A, B, C, D, E\}$ we compute ghost's probability to arrive there by adding up the probabilities of all the inbound arrows (u, v) (multiplying the previous probability of u by a coefficient

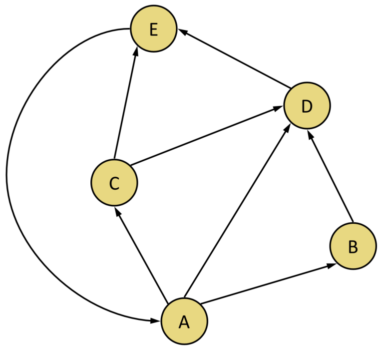


Figure 1: Arrows showing the possible moves.

1, (1/2), (1/3) (depending on how many arrows leave vertex u). Let us denote by $p_A^{(n)}$ the probability of ghost being in A (after n steps, where $n = 0, 1, 2, \ldots$). The probability of being in A at the next step is denoted as $p_A^{(n+1)}$. (Similarly for vertices B, C, D, E.)

$$\begin{cases} p_A^{(n+1)} = 1 \cdot p_E^{(n)} \\ p_B^{(n+1)} = \frac{1}{3} \cdot p_A^{(n)} \\ p_C^{(n+1)} = \frac{1}{3} \cdot p_A^{(n)} \\ p_D^{(n+1)} = \frac{1}{3} \cdot p_A^{(n)} + 1 \cdot p_B^{(n)} + \frac{1}{2} \cdot p_C^{(n)} \\ p_E^{(n+1)} = \frac{1}{2} \cdot p_C^{(n)} + 1 \cdot p_D^{(n)} \end{cases}$$

This is multiplication of a matrix with a probability vector. We multiply the initial probability vector $(1/5, 1/5, 1/5, 1/5, 1/5)$ with the same matrix one, two, and three times.

$$\begin{pmatrix} p_A^{(1)} \\ p_B^{(1)} \\ p_C^{(1)} \\ p_D^{(1)} \\ p_E^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1/5 \\ 1/5 \\ 1/5 \\ 1/5 \\ 1/5 \end{pmatrix} = \begin{pmatrix} 1/5 \\ 1/15 \\ 1/15 \\ 11/30 \\ 3/10 \end{pmatrix}.$$

$$\begin{pmatrix} p_A^{(2)} \\ p_B^{(2)} \\ p_C^{(2)} \\ p_D^{(2)} \\ p_E^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1/5 \\ 1/15 \\ 1/15 \\ 11/30 \\ 3/10 \end{pmatrix} = \begin{pmatrix} 3/10 \\ 1/15 \\ 1/15 \\ 1/6 \\ 2/5 \end{pmatrix}.$$

$$\begin{pmatrix} p_A^{(3)} \\ p_B^{(3)} \\ p_C^{(3)} \\ p_D^{(3)} \\ p_E^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 3/10 \\ 1/15 \\ 1/15 \\ 1/6 \\ 2/5 \end{pmatrix} = \begin{pmatrix} 2/5 \\ 1/10 \\ 1/10 \\ 17/60 \\ 13/60 \end{pmatrix}.$$

□

(b) Find the limit of the probabilities for the ghost to be in any of the five rooms as the number of steps $n \to \infty$.

Denote the limit values of ghost probabilities by p_A^*, p_B^*, p_C^*, p_D^*, p_E^* (and they add up to 1). In the limit they satisfy the system of linear equations:

$$\begin{cases} p_A^* & = & & & & 1 \cdot p_E^* \\ p_B^* & = & \frac{1}{3} \cdot p_A^* & & & \\ p_C^* & = & \frac{1}{3} \cdot p_A^* & & & \\ p_D^* & = & \frac{1}{3} \cdot p_A^* & +1 \cdot p_B^* & +\frac{1}{2} \cdot p_C^* & \\ p_E^* & = & & & \frac{1}{2} \cdot p_C^* & +1 \cdot p_D^* \end{cases}$$

Bring all terms from the right side to the left side (and add the 6th equation for the sum of all probabilities):

$$\begin{cases} p_A^* & & & & -p_E^* & = 0 \\ -\frac{1}{3} \cdot p_A^* & +p_B^* & & & & = 0 \\ -\frac{1}{3} \cdot p_A^* & & +p_C^* & & & = 0 \\ -\frac{1}{3} \cdot p_A^* & -p_B^* & -\frac{1}{2} \cdot p_C^* & +p_D^* & & = 0 \\ & & -\frac{1}{2} \cdot p_C^* & -p_D^* & +p_E^* & = 0 \\ p_A^* & +p_B^* & +p_C^* & +p_D^* & +p_E^* & = 1 \end{cases}$$

Solve this system with method of exclusion, find the following solution:

$$(p_A^*, p_B^*, p_C^*, p_D^*, p_E^*) = \left(\frac{2}{7}, \frac{2}{21}, \frac{2}{21}, \frac{5}{21}, \frac{2}{7} \right).$$

Note. This type of calculation is similar to Google Page Rank – it uses a similar "random ghost" model to determine which Web pages have more inbound links (and which of the inbound links come from pages that are themselves more popular). □