Homework 13

Discrete Structures
Due Friday, April 16, 2021

Submit each question separately as .pdf

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1. Recall that an undirected graph is *k-regular* if every vertex has degree k. Prove that a $2k$-regular graph has no cut edges, for every $k \in \mathbf{N}$.

2. Let $G = (V, E)$ be bipartite. Prove that G does not have C_n as a subgraph, for n odd.

3. Construct an ordered rooted tree whose postorder traversal is

$$a, c, f, g, e, b, i, j, k, n, m, o, p, \ell, h, d.$$

   In this graph the vertex ℓ has four children, b has three children, d, e, h, m have two children each, and all other vertices are leaves.

4. Let G be a graph with 100 vertices with the following property: The graph G does not contain K_3 as a subgraph. Estimate the largest possible number of edges in G.
   Note. An estimate has 2 parts. A *lower bound* shows a graph $G = (V, E)$ with the property and a possibly large number of edges $|E| = m_1$. An *upper bound* proves that for $|E| = m_2$ the property must fail. (Ideally, $m_2 = m_1 + 1$; it would be the exact estimate.)

5. A computer game uses a labyrinth – the directed graph shown in Figure 1. In the beginning a ghost enters one of the 5 rooms A, B, C, D or E (any room with the same probability $p = 0.2$). During the first step the ghost randomly chooses one of the outbound edges of its current room and moves to another room; during the next step it takes another outbound edge from its current state and so on.

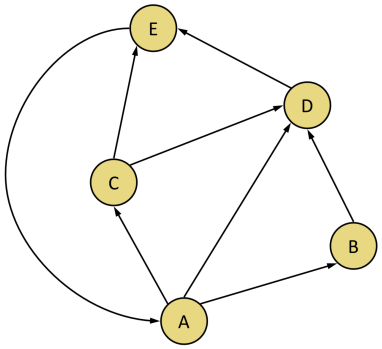

Figure 1: Arrows showing the possible moves.

   (a) Find the probabilities for every room where the ghost will be after one, two and three steps.

   (b) Find the limit of the probabilities for the ghost to be in any of the five rooms as the number of steps $n \to \infty$.