Midterm, Var.2

Discrete Structures
Thursday, April 15, 2021

You must justify all your answers to recieve full credit

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1. (a) Let x and y be propositions. Transform the following Boolean expression into an equivalent one:

$$(x \wedge (x \to y)) \to y. \quad (1)$$

Alternative 1. We simplify the expression applying one Boolean equivalence at a time:

$$\begin{aligned}
&(x \wedge (x \to y)) \to y \equiv \\
\equiv &(x \wedge (\neg x \vee y)) \to y \equiv \quad (\textit{Rewrite implication as disjunction}) \\
\equiv &((x \wedge \neg x) \vee (x \wedge y)) \to y \equiv \quad (\textit{Distributivity law for conjunction}) \\
\equiv &(\texttt{False} \vee (x \wedge y)) \to y \equiv \quad (\textit{Contradiction } (x \wedge \neg x)) \\
\equiv &(x \wedge y) \to y \equiv \quad (\textit{Disjunction with "False"}) \\
\equiv &\neg(x \wedge y) \vee y \equiv \quad (\textit{Rewrite implication as disjunction}) \\
\equiv &(\neg x \vee \neg y) \vee y \equiv \quad (\textit{De Morgan's Law}) \\
\equiv &\neg x \vee (\neg y \vee y) \equiv \quad (\textit{Associativity of Disjunction}) \\
\equiv &\neg x \vee \texttt{True} \equiv \quad (\textit{Law of Excluded Middle}) \\
\equiv &\texttt{True} \quad (\textit{Disjunction with "True"})
\end{aligned}$$

Alternative 2. If this seems to long, you can also use truth tables (and then find a shorter description of the formula):

x	y	$(x \to y)$	$(x \wedge (x \to y))$	$(x \wedge (x \to y)) \to y$
`False`	`False`	`True`	`False`	`True`
`False`	`True`	`True`	`False`	`True`
`True`	`False`	`False`	`False`	`True`
`True`	`True`	`True`	`True`	`True`

The last column reads that the expression is equivalent to `False`. □

(b) Write an equivalent Boolean expression – the contrapositive of the implication (1).

Rewrite $(x \wedge (x \to y)) \to y$ as contrapositive:

$$\neg y \to \neg(x \wedge (x \to y)).$$

□

Note. You may need to use the Boolean equivalences, *Chapter* 1.3.2 in (Rosen2019, p.29).

2. A Boolean expression $f(x, y, z)$ is `True` iff exactly two of its 3 arguments are `True`.

(a) Create the truth table for the expression $f(x, y, z)$.

There are just three combinations of (x, y, z) where exactly two `True` and one `False`.

x	y	z	$f(x, y, z)$
`False`	`False`	`False`	`False`
`False`	`False`	`True`	`False`
`False`	`True`	`False`	`False`
`False`	`True`	`True`	`True`
`True`	`False`	`False`	`False`
`True`	`False`	`True`	`True`
`True`	`True`	`False`	`True`
`True`	`True`	`True`	`False`

□

(b) Find the Disjunctive Normal Form (DNF) for the expression $f(x, y, z)$.

For each line where $f(x, y, z)$ is `True` we write one term in DNF.

$$(\neg x \wedge y \wedge z) \vee (x \wedge \neg y \wedge z) \vee (x \wedge y \wedge \neg z).$$

□

3. Let U be some universe and A, B, C are three subsets of it. Consider the following set expression:

$$S = ((B - A) \cup (B - C)) \oplus (A \cap C).$$

Draw a Venn diagram for these sets and shade all the regions corresponding to the set S. *Note.* To show your work also draw some intermediate results for this set expression.

We build the set S step by step (the ultimate Venn diagram is in the right bottom corner of Figure 1).

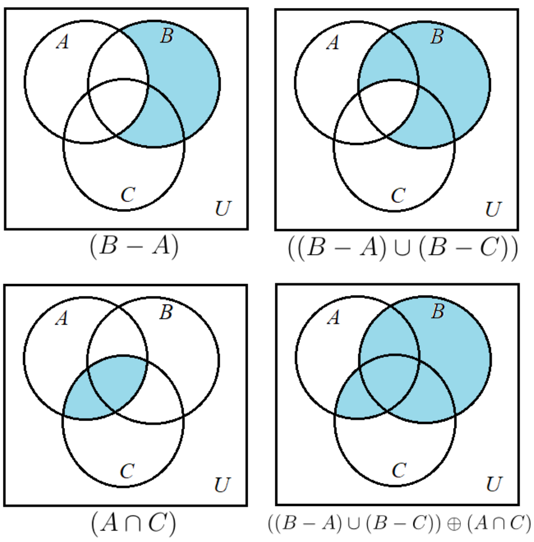

Figure 1: Venn diagram for some set expressions.

□

4. Consider these statements:
Statement1: *"Some people are over 70, but not all people in health-risk groups are older than 70."*
Statement2: *"For anyone who is over 70 or in a health-risk group there is some appropriate vaccine."*

Let us introduce two sets: H – the set of all humans; V – the set of all vaccines.
You can use the following predicates:
$S(h)$ is true iff the human h is a senior citizen (over 70).
$R(h)$ is true iff the human h is in a health risk group.
$A(v, h)$ is true iff the vaccine v is appropriate for the human h.

Translate both statements into predicate logic.

Statement 1: $\exists h \in H\ (S(h))\ \wedge\ \exists h \in H\ (R(h) \wedge \neg S(h))$.
Note. Both variables named h are in different scopes, so they are unrelated. If you wish, you can rename the variable in both scopes:

$$\exists h_1 \in H\ (S(h_1))\ \wedge\ \exists h_2 \in H\ (R(h_2) \wedge \neg S(h_2)).$$

Statement 2: $\forall h \in H\ \exists v \in V\ (S(h) \vee R(h) \rightarrow A(v, h))$. □

5. Check the following nested predicate statement on the predicates defined below:

$$\forall x, y \in \mathbf{N}\ \exists z \in \mathbf{N}\ ((x < y) \rightarrow P(x) \rightarrow P(y) \rightarrow (\neg Q(z) \wedge x \leq z \wedge z \leq y)). \qquad (2)$$

$P(x)$ defined on $\mathbf{N} = \{0, 1, 2, \ldots\}$; it is `True` iff x is full square.
$Q(x)$ defined on $\mathbf{N}$; it is true iff the last digit of x is either 1 or 2.

Determine whether the predicate expression (2) is `True` or `False`; explain your answer.
Note. Both predicates are defined on the infinite set $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ of natural

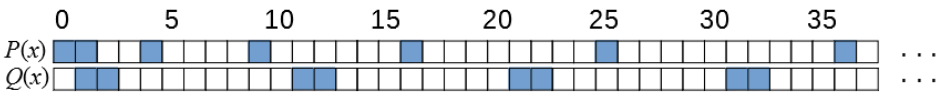

Figure 2: Predicate values that equal to `True` are shaded; `False` are white.

numbers. Some initial values are shown in Figure 2.

Answer. The expression has value `True`.

Logical implication is right-associative, so the sub-expression

$$(x < y) \rightarrow P(x) \rightarrow P(y) \rightarrow (\neg Q(z) \wedge x \leq z \wedge z \leq y)$$

can be rewritten like this:

$$(x < y) \rightarrow (P(x) \rightarrow (P(y) \rightarrow (\neg Q(z) \wedge x \leq z \wedge z \leq y))).$$

The only case when this long implication could be `False`, is the case when all the conditions $((x < y)$, $P(x)$ and $P(y))$ are `True`, but the conclusion $((\neg Q(z) \wedge x \leq z \wedge z \leq y))$ is `False`.

Assume that x, y are two non-negative integers $x < y$ and $P(x)$, $P(y)$ (so both x and y are full squares).
Can we always find a number z such that $\neg Q(z)$ (z does not end with digit 1 or 2) and $x \leq z \leq y$? Numbers that end with digit 1 or digit 2 come in pairs (1 and 2, or 11 and 12, or 21 and 22, etc.) The only situation where all integers $z \in [x; y]$ make $Q(x)$ true, would be when both x and y are in one such pair (i.e. either $x = 1, y = 2$ or $x = 11, y = 12$ etc.). This is impossible since no full square ends with a digit 2. Therefore $(\neg Q(z) \wedge x \leq z \wedge z \leq y)$ can be always satisfied and the entire expression is `True`. □

6. Use the set-builder notation to describe the set S of all non-negative integers that can be expressed as the sum of an even and odd square in just one way (the order of addition does not matter). Here are some examples of numbers belonging to S:

$$\begin{aligned} 5 &= 2^2 + 1^2, \\ 13 &= 3^2 + 2^2, \\ 17 &= 4^2 + 1^2, \\ 29 &= 5^2 + 2^2, \\ 37 &= 6^2 + 1^2, \\ 41 &= 5^2 + 4^2. \end{aligned}$$

In the set-builder notation you can use $\mathbf{N}$ (natural numbers, i.e. the set of all non-negative integers), arithmetic operations, Boolean operations, quantifiers, congruence notation ($a \equiv b \pmod{m}$), equals, not equals and other inequalities (i.e. the predicates $a = b$, $a \neq b$, $a < b$, $a \leq b$).

Note. Please observe that, for example, $25 \not\in S$, since $25 = 5^2 + 0^2 = 4^2 + 3^2$ (therefore, 25 can be expressed as a sum of an even and an odd square in more than one way).

The set-builder notation for the set S:

$$S = \{x \in \mathbf{N} \mid x \equiv 1 \pmod{2} \;\wedge\; \exists a \in \mathbf{N}\; \exists b \in \mathbf{N}$$

$$((a < b) \;\wedge\; a^2 + b^2 = x \;\wedge\; \forall c \in \mathbf{N}\; \forall d \in \mathbf{N}\; ((c < d) \rightarrow c^2 + d^2 = x \rightarrow (a = c \wedge b = d)))\}.$$

This expression tells the following: x is an odd number (this automatically means that a^2 and b^2 cannot be both odd or both even); there exist two natural numbers $a < b$ with $a^2 + b^2 = x$; and for any other two natural numbers $c < d$ such that $c^2 + d^2 = x$ we must have $(a, b) = (c, d)$. □

7. Define the following function $f(x)$ for real x:

$$f(x) = \begin{cases} x + 1, & \text{if } \lfloor x \rfloor \equiv 0 \pmod{3} \\ x - 1, & \text{if } \lfloor x \rfloor \equiv 1 \pmod{3} \\ x, & \text{if } \lfloor x \rfloor \equiv 2 \pmod{3} \end{cases}$$

   (a) Identify what is the domain, the codomain and the range of the function f (assuming we want to use it for all those arguments where this expression makes sense).

   Domain is $\mathbf{R}$ (it is defined for all real numbers; any real number has $\lfloor x \rfloor$ being congruent to 0, 1 or 2 (modulo 3). Codomain is also $\mathbf{R}$ as $f(x)$ takes real values.

The range of $f(x)$ by the definition is the set of all those values y in the codomain that have some x such that $f(x) = y$. We claim that the range of $f(x)$ is also the entire set $\mathbf{R}$.

The three clauses in the definition of this function "mix" the half-open intervals $[n; n+1)$ in such a way that the interval $x \in [0; 3)$ maps to $f(x) \in [0; 3)$; also $x \in [3; 6)$ maps to $f(x) \in [3; 6)$ and so on. See the graph of this function in Figure 3.

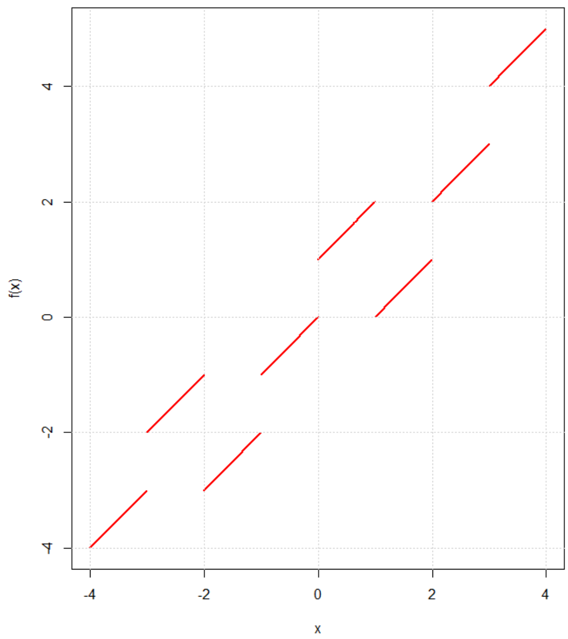

Figure 3: Graph for the $f(x)$.

□

(b) Is function f injective? surjective? bijective?

The function is injective and surjective (and thus bijective). It just swaps the order of unit-length intervals in $[0; 3)$, in $[3; 6)$ and so on.

□

8. Express the following sum as a closed formula:

$$f(n) = \sum_{k=1}^{n} \left\lfloor \frac{k}{2} \right\rfloor .$$

Note. By a closed formula we mean an expression which can be used to compute this sum without using summation sumbols, recurrent sequences or expressions with ....

Let us first sum the arithmetic series (same sequence without the floor function):

$$\sum_{k=1}^{n} \frac{k}{2} = \frac{1}{2} + \frac{2}{2} + \ldots + \frac{n}{2} = \left(\frac{1}{2} + \frac{n}{2} \right) : 2 \cdot n = \frac{n+1}{4} \cdot n.$$

Next, observe that the floor function subtracts $1/2$ from the first, the third, the fifth term and so on. For any given value n, the number of halves that one needs to subtract can be expressed as $\lfloor (n+1)/2 \rfloor$. This has to be multiplied by $1/2$ (because we subtract one extra $1/2$ every time we reach the next odd value n). We get the following:

$$f(n) = \frac{n+1}{4} \cdot n - \left\lfloor \frac{n+1}{2} \right\rfloor \cdot \frac{1}{2} = \frac{n^2+n}{4} - \frac{1}{2} \left\lfloor \frac{n+1}{2} \right\rfloor .$$

□

9. Is the number $\log_4 8 + \sqrt[4]{8}$ rational or irrational?

   Note. Justify your answer. For example, express it as a rational fraction P/Q (if it is rational) or get a contradiction assuming that it is rational (if it is irrational).

10. Find the least integer n such that $f(x)$ is $O(x^n)$ for the function $f(x) = 2x^3 + x^2 \log_2 x$. Justify your answer.

11. Show that x^3 is in $O(x^4)$, but x^4 is not in $O(x^3)$.

12. Consider numbers $a = 171$, $b = 132$.

   (a) Use Euclidean algorithm to find $d = \gcd(a, b)$, the greatest common divisor of these numbers. Show intermediate steps.

   Th Euclidean algorithm runs as follows:

$$\begin{array}{rcrcr} 171 & = & 1 \cdot 132 & + & 39 \\ 132 & = & 3 \cdot 39 & + & 15 \\ 39 & = & 2 \cdot 15 & + & 9 \\ 15 & = & 1 \cdot 9 & + & 6 \\ 9 & = & 1 \cdot 6 & + & 3 \\ 6 & = & 2 \cdot 3 & + & 0 \end{array}$$

   The gcd is the last nonzero remainder, which is 3. □

   (b) For the GCD number d found in the previous step, find an integer solution (x, y) to the Bezout identity:

$$ax + by = d.$$

   Show the steps how x and y were obtained.

   We work upwards, starting with the second-to-last equation, and replacing equation above inside it:

$$\begin{array}{rll} 9 = 1 \cdot 6 + 3 & & \\ 9 = 1 \cdot (15 - 1 \cdot 9) + 3 & & \text{(replace)} \\ 2 \cdot 9 = 1 \cdot 15 + 3 & & \text{(simplify)} \\ 2 \cdot (39 - 2 \cdot 15) = 1 \cdot 15 + 3 & & \text{(replace)} \\ 2 \cdot 39 - 5 \cdot 15 = 3 & & \text{(simplify)} \\ 2 \cdot 39 - 5 \cdot (132 - 3 \cdot 39) = 3 & & \text{(replace)} \\ 17 \cdot 39 - 5 \cdot 132 = 3 & & \text{(simplify)} \\ 17 \cdot (171 - 1 \cdot 132) - 5 \cdot 132 = 3 & & \text{(replace)} \\ 17 \cdot 171 - 22 \cdot 132 = 3 & & \text{(simplify)} \end{array}$$

   Hence $a = 17$ and $b = -22$. □

13. The following number is written in binary representation (its total length is 99 binary digits):

$$N = \underbrace{101101\cdots101_2}_{\text{33 times repeat same three digits "101"}}.$$

(a) Express the value of N with a closed formula (without $\cdots$); add it up using some summation formula for geometric series.

First note that $101_2 = 5 = 2^2 + 2^0$, and $101000_2 = 40 = 2^3 \cdot 5$. This pattern generalizes, hence

$$\sum_{n=0}^{32} 5 \cdot (2^3)^n = 5 \cdot \frac{1 - 8^{33}}{1 - 8} = \frac{5(1 - 8^{33})}{7}, \tag{3}$$

using the partial sum geometric series formula for a ration $r = 2^3 = 8$ and a first term $a = 5$. □

(b) What is the length of N in octal representation (base 8), hexadecimal representation (base 16) and decimal representation (base 10)?

For N the value computed at line (3), the length of each of these expressions, in terms of the number of digits is:

$$\begin{aligned} \text{octal}: &\quad \lceil \log_8(N) \rceil \\ \text{hexadecimal}: &\quad \lceil \log_{16}(N) \rceil \\ \text{decimal}: &\quad \lceil \log_{10}(N) \rceil \end{aligned}$$

□

14. Consider the following rational fraction: $r = \frac{123}{192}$.

(a) Express r as a binary fraction.

As a binary fraction, this is $\dfrac{1111011_2}{11000000_2}$. □

(b) Express r as a hexadecimal fraction.

As a hexadecimal fraction, this is $\dfrac{7B_{16}}{C0_{16}}$. □

15. Prove by mathematical induction that for all $n \in \mathbf{Z}^+$ the following inequality holds:

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{8}\right)\cdots\left(1 - \frac{1}{2^n}\right) \geq \frac{1}{4} + \frac{1}{2^{n+1}}.$$