Midterm, Var.2

Discrete Structures
Thursday, April 15, 2021

You must justify all your answers to recieve full credit

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1. (a) Let x and y be propositions. Transform the following Boolean expression into an equivalent one:
$$(x \wedge (x \to y)) \to y. \tag{1}$$
   (b) Write an equivalent Boolean expression – the contrapositive of the implication (1).

   Note. You may need to use the Boolean equivalences, *Chapter* 1.3.2 in (Rosen2019, p.29).

2. A Boolean expression $f(x, y, z)$ is `True` iff exactly two of its 3 arguments are `True`.
   (a) Create the truth table for the expression $f(x, y, z)$.
   (b) Find the Disjunctive Normal Form (DNF) for the expression $f(x, y, z)$.

3. Let U be some universe and A, B, C are three subsets of it. Consider the following set expression:
$$S = ((B - A) \cup (B - C)) \oplus (A \cap C).$$
   Draw a Venn diagram for these sets and shade all the regions corresponding to the set S.
   Note. To show your work also draw some intermediate results for this set expression.

4. Consider these statements:
   Statement1: *"Some people are over 70, but not all people in health-risk groups are older than 70."*
   Statement2: *"For anyone who is over 70 or in a health-risk group there is some appropriate vaccine."*

   Let us introduce two sets: H – the set of all humans; V – the set of all vaccines.
   You can use the following predicates:
   $S(h)$ is true iff the human h is a senior citizen (over 70).
   $R(h)$ is true iff the human h is in a health risk group.
   $A(v, h)$ is true iff the vaccine v is appropriate for the human h.

   Translate both statements into predicate logic.

5. Check the following nested predicate statement on the predicates defined below:
$$\forall x, y \in \mathbf{N}\ \exists z \in \mathbf{N}\ ((x < y) \to P(x) \to P(y) \to (\neg Q(z) \wedge x \leq z \wedge z \leq y)). \tag{2}$$
   $P(x)$ defined on $\mathbf{N} = \{0, 1, 2, \ldots\}$; it is `True` iff x is full square.
   $Q(x)$ defined on $\mathbf{N}$; it is true iff the last digit of x is either 1 or 2.

   Determine whether the predicate expression (2) is `True` or `False`; explain your answer.
   Note. Both predicates are defined on the infinite set $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ of natural numbers. Some initial values are shown in Figure 1.

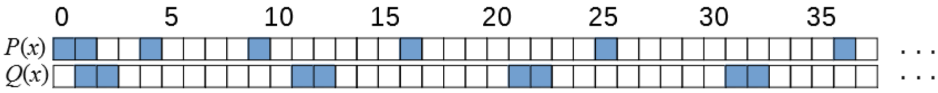

Figure 1: Predicate values that equal to `True` are shaded; `False` are white.

6. Use the set-builder notation to describe the set S of all non-negative integers that can be expressed as the sum of an even and odd square in just one way (the order of addition does not matter). Here are some examples of numbers belonging to S:

$$\begin{aligned} 5 &= 2^2 + 1^2, \\ 13 &= 3^2 + 2^2, \\ 17 &= 4^2 + 1^2, \\ 29 &= 5^2 + 2^2, \\ 37 &= 6^2 + 1^2, \\ 41 &= 5^2 + 4^2. \end{aligned}$$

In the set-builder notation you can use $\mathbf{N}$ (natural numbers, i.e. the set of all non-negative integers), arithmetic operations, Boolean operations, quantifiers, congruence notation ($a \equiv b \pmod{m}$), equals, not equals and other inequalities (i.e. the predicates $a = b$, $a \neq b$, $a < b$, $a \leq b$).

Note. Please observe that, for example, $25 \not\in S$, since $25 = 5^2 + 0^2 = 4^2 + 3^2$ (therefore, 25 can be expressed as a sum of an even and an odd square in more than one way).

7. Define the following function $f(x)$ for real x:

$$f(x) = \begin{cases} x+1, & \text{if } \lfloor x \rfloor \equiv 0 \pmod{3} \\ x-1, & \text{if } \lfloor x \rfloor \equiv 1 \pmod{3} \\ x, & \text{if } \lfloor x \rfloor \equiv 2 \pmod{3} \end{cases}$$

   (a) Identify what is the domain, the codomain and the range of the function f (assuming we want to use it for all those arguments where this expression makes sense).

   (b) Is function f injective? surjective? bijective?

8. Express the following sum as a closed formula:

$$f(n) = \sum_{k=1}^{n} \left\lfloor \frac{k}{2} \right\rfloor.$$

Note. By a closed formula we mean an expression which can be used to compute this sum without using summation sumbols, recurrent sequences or expressions with $\ldots$.

9. Is the number $\log_4 8 + \sqrt[4]{8}$ rational or irrational?

Note. Justify your answer. For example, express it as a rational fraction P/Q (if it is rational) or get a contradiction assuming that it is rational (if it is irrational).

10. Find the least integer n such that $f(x)$ is $O(x^n)$ for the function $f(x) = 2x^3 + x^2 \log_2 x$. Justify your answer.

11. Show that x^3 is in $O(x^4)$, but x^4 is not in $O(x^3)$.

12. Consider numbers $a = 171$, $b = 132$.

(a) Use Euclidean algorithm to find $d = \gcd(a, b)$, the greatest common divisor of these numbers. Show intermediate steps.

(b) For the GCD number d found in the previous step, find an integer solution (x, y) to the Bezout identity:

$$ax + by = d.$$

Show the steps how x and y were obtained.

13. The following number is written in binary representation (its total length is 99 binary digits):

$$N = \underbrace{101101\cdots101_2}_{\text{33 times repeat same three digits "101"}}.$$

(a) Express the value of N with a closed formula (without $\cdots$); add it up using some summation formula for geometric series.

(b) What is the length of N in octal representation (base 8), hexadecimal representation (base 16) and decimal representation (base 10)?

14. Consider the following rational fraction: $r = \frac{123}{192}$.

(a) Express r as a binary fraction.

(b) Express r as a hexadecimal fraction.

15. Prove by mathematical induction that for all $n \in \mathbf{Z}^+$ the following inequality holds:

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{8}\right)\cdots\left(1 - \frac{1}{2^n}\right) \geq \frac{1}{4} + \frac{1}{2^{n+1}}.$$