Midterm

Discrete Structures
Wednesday, February 24, 2021

You must justify all your answers to recieve full credit

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1. (a) Write the following sentence as a Boolean expression: "To get an early vaccine it is sufficient to be a public servant involved in the continuous operation of government or a senior citizen with a referral from a family doctor."
   Use the following atomic propositions:
   V := "One can get an early vaccine"
   G := "One is a public servant involved in the continuous operation of government"
   S := "One is a senior citizen"
   R := "One has a referral from a family doctor."

   (b) Write an equivalent Boolean expression – the contrapositive of the previous one.

2. Let A, B, C be subsets in the same universe U. Draw a Venn diagram for these sets and shade all the regions corresponding to the set S:

$$S = (A \cup B \cup \overline{C}) \cap (A \cup \overline{B} \cup C) \cap (\overline{A} \cup B \cup C).$$

3. Let A, B, C be three arbitrary subsets of the same universe U. Prove or disprove the following set identity:

$$(B \oplus C) \cup A = (B \cup C) \oplus (A - C).$$

4. Let $P(x, y)$ and $Q(x, y)$ be two predicates defined on pairs of integers. Simplify the expression so that all negations are applied directly to the predicate symbols:

$$\neg(\forall y \in \mathbf{Z}\ (\neg Q(x, z) \vee P(x, y)) \wedge \exists z \in \mathbf{Z}\ \forall x \in \mathbf{Z}\ (Q(y, z) \to \neg P(x, y))).$$

5. Simplify the expression with negation so that negation is only applied to individual predicates or propositional variables (rather than larger subexpressions or quantifiers):

6. Use the set-builder notation to describe the set of all positive odd integers n such that for every prime p dividing n, the number p^2 also divides n. Here is an (incomplete) list of the numbers in this set:

$$S = \{1, 9, 25, 27, 49, 81, 121, 125, 169, 225, 243, \ldots\}.$$

   In the set-builder notation you can use $\mathbf{Z}^+$ (all positive integers), arithmetic operations, Boolean operations, quantifiers, and these two predicates:
   Prime(x) is true iff x is a prime.
   $(a \mid b)$ is true iff a divides b.

7. Prove or disprove by a counterexample the following two statements:

   (a) Statement$_1$: "Function $f\colon \mathbf{R} \to \mathbf{R}$ given by $f(x) = 3x - 7$ is surjective."

   (b) Statement$_2$: "Function $f\colon \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = 3x - 7$ is surjective."

8. Is the number $\dfrac{2}{1+\sqrt{5}}$ rational or irrational? Prove your answer.
(If necessary, you can use the following Lemma: For any positive integer n, the square root $\sqrt{n}$ is either itself a positive integer or it is irrational.)

9. Consider set S defined by this set-builder expression:

$$S = \left\{ x \in \mathbf{Z}^+ \mid x \leqslant 80 \land \exists m \in \mathbf{Z}^+ \; \left(x = m^2 \right) \right\}.$$

   (a) List the elements of the set S.
   (b) Find the size of its power set $|\mathcal{P}(S)|$.

10. Let $f\colon \mathbf{N} \to \mathbf{N}$ be defined by $f(n) = \sum\limits_{j=1}^{n} j(j+1)$.

   (a) Find the smallest k such that $f(n)$ is in $O(n^k)$.
   (b) Find C, n_0 so that $|f(n)|$ does not exceed $C \cdot |n^k|$ for all $n \geqslant n_0$.

11. (a) Use Euclidean algorithm to find $\gcd(426, 156)$ (*the greatest common divisor*).
   (b) Use the GCD found in the previous step to compute $\mathrm{lcm}(426, 156)$ (*the least common multiple*).

12. Consider the system of congruences

$$\begin{cases} x \equiv 1 \pmod{5}, \\ x \equiv 2 \pmod{7}, \\ x \equiv 3 \pmod{9}. \end{cases}$$

   (a) Find one solution to this system of congruences.
   (b) Describe all the solutions to this system.

13. Consider the two numbers in binary notation

$$\alpha = 111001110_2,$$
$$\beta = 1110_2.$$

   (a) Express β as a sum of powers of 2.
   (b) Show how to multiply the two binary numbers α and β on paper (similar to the "school algorithm". It would look like this – with 0s and 1s instead of asterisks:

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 111001110
 ×     1110
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 *********
*********
...   ...
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14. Express the periodic decimal fraction $3.378378378\ldots = 3.(378)$ as an irreducible rational number $\frac{p}{q}$. Show the formulas (infinite geometric progression or some other arithmetic manipulation) that can be used to get your answer.

15. Prove by mathematical induction that for all $n \in \mathbf{Z}^+$ the following equality holds:

$$\sum_{j=1}^{n} \left(\frac{1}{2j-1} - \frac{1}{2j+1} \right) = \frac{2n}{2n+1}.$$