Worksheet 2

2.a. For all natural numbers x, dividing x by 1 leaves a remainder of 0.

2.b. There exists a real number z such that for all real numbers x, dividing x by y does not leave a remainder of z.

2.c. For all integers x, if x is greater that or equal to 1 and dividing 1 by x leaves a remainder of z, then either z is 1 or z is 0.

2.d. For all real numbers x_1 and all nonzero real numbers x_2, there exist real numbers a and b such that dividing x_1 by x_2 leaves a remainder z that is strictly between a and b.

3.a. Not valid. $P \to Q$ was given, but $Q \to P$ was concluded, which is not equivalent.

3.b. Valid. $P \to Q$ was given, and $\neg Q \to \neg P$ was concluded, which is the contrapositive.

3.c. Not valid. $P \to Q$ was given, but $\neg P \to \neg Q$ was concluded, which is not equivalent.

3.d. Valid. $P \to Q$ was given and $P \to Q \to Q_2 \to Q_3$ was concluded, with $Q \to Q_2$ and $Q_2 \to Q_3$ of the same form as $P \to Q$:

$P: n > 0$

$Q: n+1 > 0$

$Q_2: (n+1)+1 = n+2 > 0$

$Q_3: (n+2)+1 = n+3 > 0$

4.a. $\neg(P \vee Q) = \neg P \wedge \neg Q$

$=$ You are not rich and not happy.

4.b. $\neg(\exists x\, P(x)) = \forall x\, \neg P(x)$

$=$ Every person is younger than 120

4.c. $\neg(P \rightarrow Q) = P \wedge \neg Q$

*Careful: Q is the first part of the given statement.

$=$ $3x-1$ is a perfect square and the square root of $2x+1$ is not an integer.

$\neg Q \rightarrow \neg P$ $=$ If the square root of $2x+1$ is not an integer, then $3x-1$ is not a perfect square.

4.d. $\neg\left(P \rightarrow \left(\exists \varepsilon \geq 0\left((Q_1 \wedge Q_2) \vee (Q_3 \wedge Q_4)\right)\right)\right)$

$= P \wedge \neg\left(\exists \varepsilon \geq 0\left((Q_1 \wedge Q_2) \vee (Q_3 \wedge Q_4)\right)\right)$

$= P \wedge \forall \varepsilon \geq 0\left(\neg(Q_1 \wedge Q_2) \wedge \neg(Q_3 \wedge Q_4)\right)$

$= P \wedge \forall \varepsilon \geq 0\left((\neg Q_1 \vee \neg Q_2) \wedge (\neg Q_3 \vee \neg Q_4)\right)$

= A function has a root at $x=a$ and for all $\varepsilon > 0$ both $f(a-\varepsilon) \leq 0$ or $f(a+\varepsilon) \geq 0$, and $f(a-\varepsilon) \geq 0$ or $f(a+\varepsilon) \leq 0$.

5.a. $\neg(P \to Q) = P \wedge \neg Q$

= I am older than 21 and younger than 18.

5.b. $\neg(\forall x\, P(x)) = \exists x\, \neg P(x)$

= There exists a person living in Chicago that was not born in Chicago.

5.c. $\neg(P \to (Q \vee R \vee S)) = P \wedge \neg(Q \vee R \vee S)$

$= P \wedge (\neg Q \wedge \neg R \wedge \neg S)$

= Both a, b are natural numbers and $\frac{a}{b}$ is not less than one, not equal to one, and not greater than one.

* there was a mistake here: "$\frac{a}{b} = 1$ or $\frac{a}{b} > 1$"

5.d. $\neg(P \to Q) \equiv P \wedge \neg Q$

$\equiv$ n is even and $3n+6$ is not even.

6. Since $x, y \in \mathbb{R}$, $x-y \in \mathbb{R}$.

Since the square is always positive, $(x-y)^2 \geq 0$

Equivalently, $x^2 - 2xy + y^2 \geq 0$

$$x^2 + y^2 \geq 2xy$$

$$\frac{x^2+y^2}{xy} \geq 2$$

This is allowed because $x>0, y>0 \Rightarrow xy>0$.

$$\frac{x^2}{xy} + \frac{y^2}{xy} \geq 2$$

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

7.a. $\forall y \in \mathbb{Z}\ \Big(P(x,y) \to \big((y=\pm 1) \vee (y=\pm x)\big)\Big)$

7.b. For all integers a, b, c, if x is prime and $ax=bc$, then either x divides b or x divides c.

7.c. Assume $R(x)$ and $P(y^2,x)$. Since $P(y^2,x)$ is true, there exists $q \in \mathbb{Z}$ with $y^2 = qx$

$$y \cdot y = q \cdot x.$$

Using $a = q$, $b = y$, $c = y$, the statement $S(x)$ gives

$$\big(R(x) \wedge (qx = y^2)\big) \to \big(P(y,x) \vee P(y,x)\big)$$

The conclusion is equivalent to $P(y,x)$.

7.d. The integers x, y have no common factors and $\sqrt{2} = \frac{x}{y}$.

7.e. Suppose $Q(x,y)$ and $\sqrt{2} = \frac{x}{y} \Leftrightarrow \sqrt{2}\,y = x \Leftrightarrow 2y^2 = x^2$. (1)

Using $S(2)$ with $a = y^2$, $b = x$, $c = x$ we get $P(x,2)$.

That is, there exists $q \in \mathbb{Z}$ with $x = 2q$.

Then equation (1) becomes $2y^2 = (2q)^2 = 4q^2$, or

$y^2 = 2q^2$.

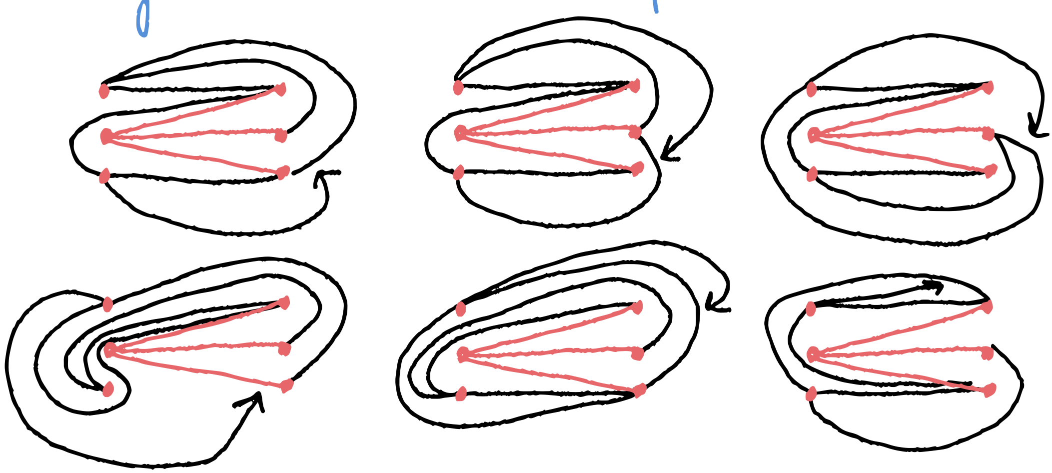
Again using $S(2)$, but with $a=q^2$, $b=y$, $c=y$, we get $P(y,2)$.

Now $P(x,2)$ and $P(y,2)$ are true, but so is $Q(x,y)$.

This is a contradiction, as $2 \neq \pm 1$.

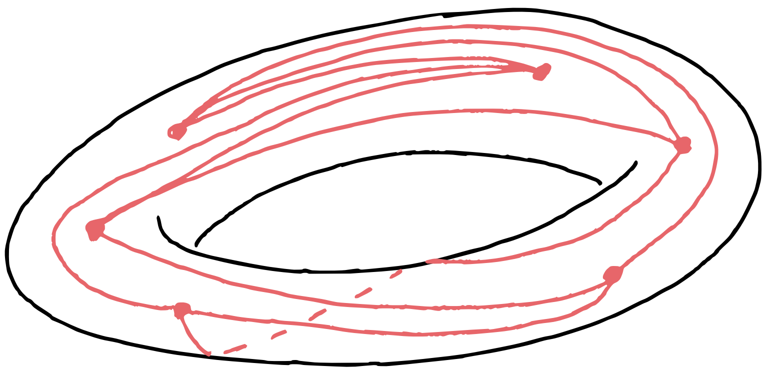
Hence $U(x,y)$ is false for all integers x, y.

8.a. By exhaustion, we try all possible sequences of adding lines. Below are some possibilities.



We have exhausted our patience - no solution exists!!

8.b. Below is one such option.



9.a. $x = zy$

9.b. $U(x, 1, 1)$

9.c. $U(x, y, 1)$

9.d. $\exists z \in \mathbb{Z}^+ \; U(x, y, z)$

9.e. $\exists y \in \mathbb{Z}^+ \; U(x, y, y)$

9.f. $\forall w \in \mathbb{Z}^+ \left(\left(\text{Divides}(w, x) \wedge \text{Divides}(w, y) \right) \rightarrow (w \leq z) \right)$

9.g. $\forall w \in \mathbb{Z}^+ \left(\text{Divides}(w, x) \rightarrow \left((w = 1) \vee (w = x) \right) \right)$

9.h. $\left(\exists a,b,c \in \mathbb{Z}^+ \left((a \neq b) \wedge (a \neq c) \wedge (b \neq c) \wedge \text{Divides}(a,x) \wedge \text{Divides}(b,x) \wedge \text{Divides}(c,x)\right)\right)$

$\rightarrow \text{Square}(x)$.

10.a. Irrational. Proved in the same way that $\sqrt{2}$ is irrational.

10.b. Irrational. Proved in (almost) the same way that $\sqrt{2}$ is irrational: $\sqrt{6} = \frac{x}{y} \Rightarrow 6y^2 = x^2$

$\Rightarrow (2 \text{ divides } x^2) \text{ or } (3 \text{ divides } x^2)$

$\vdots \qquad \vdots$

10.d. Irrational. $\sqrt[3]{7} = \frac{x}{y} \Rightarrow 7y^3 = x^3 \Rightarrow (7 \text{ divides } x^3)$

$\Rightarrow (7 \text{ divides } x) \ldots$

11a. False. $\sqrt{2} = x$ is irrational, $-\sqrt{2} = y$ is irrational,

$x + y = \sqrt{2} + (-\sqrt{2}) = 0$ is rational.

11b. False. Same x, y as above.

11.c. False. $(2^{1/3})^2 = x^2$ is irrational, $2 = (2^{1/3})^3 = x^3$ is not.