Worksheet 3

2.a. $A \cup B = \{1, 8, 7, 2, 4, 3, 6\}$

2.b. $C - B = \{7, 5, 9\}$

2.c. $C \cap A = \{7, 4\}$

2.d. $(A \cup C) - (B \cap C) = \{1, 8, 2, 7, 4\} - \{4, 6\}$

$= \{1, 8, 2, 7\}$

2.e. $A \cap B \cap C = \{4\}$

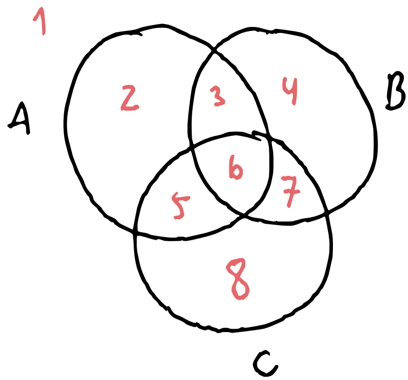
2.f. $(A \cap B \cap C) - C = \{4\} - \{7, 4, 6, 5, 9\}$

$= \emptyset$

2.g. $\overline{A \cup B} = \{5, 9\}$

2.h. $\bar{A} \cup \bar{B} = \{1, 8, 7, 5, 9, 6, 3\}$

3. We will use Venn diagrams. Below is a Venn diagram with all regions labeled.



Then we compute: $\overline{A \cup B}$ is 1, 8

$\overline{B \cup C}$ is 1, 2

$\overline{A \cup C}$ is 1, 4

Hence $\overline{A \cup B} \cap \overline{B \cup C} \cap \overline{A \cup C}$ is 1.

Similarly, we see $\bar{A}$ is 1, 4, 7, 8

$\bar{B}$ is 1, 2, 5, 8

$\bar{C}$ is 1, 2, 3, 4

Hence $\bar{A} \cap \bar{B} \cap \bar{C}$ is 1. So both are the same.

4. We do everything together.

	A_i	$\bigcup_{i=1}^{\infty} A_i$	$\bigcap_{i=1}^{\infty} A_i$
4.a.	$\{i, i+1, \ldots\}$	$\mathbb{N}$	$\emptyset$
4.b.	$\{0, i\}$	$\mathbb{N} \cup \{0\}$	$\{0\}$
4.c.	$\{-i, i\}$	$\mathbb{Z} \setminus \{0\}$	$\emptyset$
4.d.	$(0, i)$	$(0, \infty)$	$(0, 1)$
4.e.	$[-i, i]$	$\mathbb{R}$	$[-1, 1]$
4.f.	(i, ∞)	$(1, \infty)$	$\emptyset$
4.g.	$[i, \infty)$	$[1, \infty)$	$\emptyset$
4.h.	$\{-i, \ldots, i\}$	$\mathbb{Z}$	$\{-1, 0, 1\}$

Recall that $\mathbb{N} = \{1, 2, 3, \ldots\}$. All intervals are considered as subsets of the real numbers $\mathbb{R}$.

5.a.i. $\forall x \in A \left(\forall y \in A \left(\left(f(x) = f(y) \right) \rightarrow x = y \right) \right)$

5.a.ii. $\forall z \in B \left(\exists x \in A \left(f(x) = z \right) \right)$

5.a.iii. $\exists z \in B \left(\forall x \in A \left(\neg \left(f(x) = z \right) \right) \right)$

5.a.iv. $\exists z \in B \left(\exists x, y, w \in A \left(\left(x \neq y \neq w \neq x \right) \wedge \left(f(x) = f(y) = f(w) = z \right) \right) \right)$

Here the expression "$\exists x, y, w \in A$" is actually three separate existential expressions "$\exists x \in A (\exists y \in A (\exists w \in A \ldots$"

Similarly "$x \neq y \neq w \neq x$" is shorthand for the conjunctions "$(\neg (x = y)) \wedge (\neg (y = w)) \wedge (\neg (w = x))$"

Similarly "$f(x) = f(y) = f(w) = z$" is shorthand for the conjunctions "$(f(x) = z) \wedge (f(y) = z) \wedge (f(w) = z)$".

It is allowed to shorten expressions when their meaning is unambiguous (that is, there is only one possible way to interpret the expression).

5.b.i. Assume that f is injective. Then for every $x, y \in A$, if $f(x) = f(y)$, then $x = y$. Let $x_1, y_1 \in A_1$. Since $A_1 \subseteq A$, the injectivity condition holds for x_1 and y_1, so if $f(x_1) = f(y_1)$, then $x_1 = y_1$. Since $f_1(a) = f(a)$ for every $a \in A_1$, and $x_1, y_1 \in A_1$, if $f_1(x_1) = f_1(y_1)$, then $x_1 = y_1$. Hence f_1 is injective.

5.b.ii. Assume that f_1 is surjective. Then for every $b \in B_1$ there exists $a \in A_1$ with $f_1(a) = b$. Since $B_1 = B$, this condition is equivalent to "for every $b \in B$ there exists $a \in A_1$ with $f_1(a) = b$." Since $f_1(a) = f(a)$ for every $a \in A_1$ and $A_1 \subseteq A$, this condition implies "for every $b \in B$ there exists $a \in A$ with $f(a) = b$." This means f is surjective.

* Note there was a mistake initially in 5.a.ii saying "$A_1 = A$" instead of "$B_1 = B$".

6.a. Let $a, b \in \mathbb{R}$, and suppose that $f(a) = f(b)$. By the definition of f, $f(a) = a = f(b) = b$. That is, $a = b$. Hence f is injective.

6.b. Let $a, b \in \mathbb{R}$, and suppose that $g(a) = g(b)$. By the definition of g, $g(a) = (a, 0) = g(b) = (b, 0)$. By the definition of the Cartesian product, the coordinates must be the same, so $a = b$ and $0 = 0$. Since $a = b$, g is injective.

6.c. Let $a \in \mathbb{R}$. Since $a \in \mathbb{R}$ and $0 \in \mathbb{R}$, $(a, 0) \in \mathbb{R}^2$. Observe that $K(a, 0) = a$, so there exists $x \in \mathbb{R}^2$ with $K(x) = a$. Hence K is surjective.

7. Recall that if $f: X \to Y$ is a function, its <u>inverse</u> is a function $f^{-1}: Y \to X$ such that $f^{-1}(f(x)) = x$ for all $x \in X$ and $f(f^{-1}(y)) = y$ for all $y \in Y$.

7.a. $f^{-1}: \mathbb{R} \to \mathbb{R}$, $y \mapsto \sqrt[3]{y}$ $\qquad$ $g^{-1}: \mathbb{R} \to \mathbb{R}$, $y \mapsto \frac{3y+4}{15}$

$h^{-1}: \mathbb{Z} \to \mathbb{N}$

$$m \mapsto \begin{cases} 2m & m \geq 0 \\ -2m-1 & m < 0 \end{cases}$$

7.b. f is a surjection: $y \in \mathbb{R} \Rightarrow \sqrt[3]{y} \in \mathbb{R}$ and $f(\sqrt[3]{y}) = y$.

g is a surjection: $y \in \mathbb{R} \Rightarrow \frac{3y+4}{15} \in \mathbb{R}$ and $g\left(\frac{3y+4}{15}\right) = y$

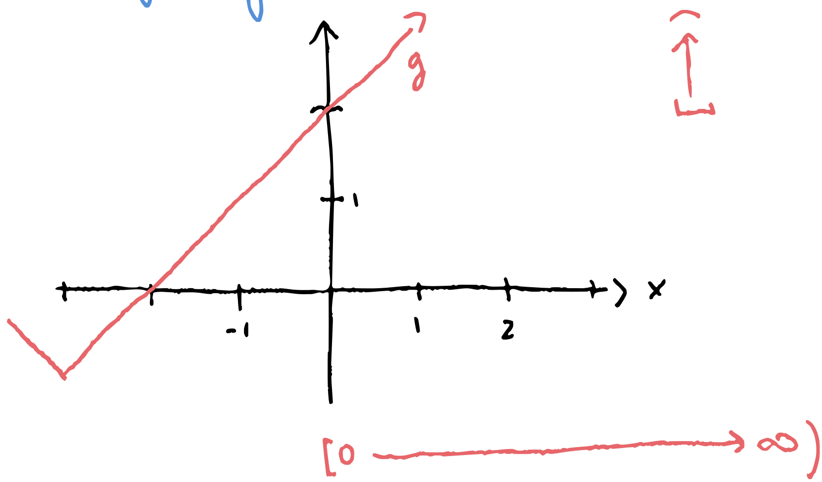
h is a surjection: $m \in \mathbb{Z}$ and $m \geq 0 \Rightarrow 2m \in \mathbb{N}$ and $h(2m) = m$

$m \in \mathbb{Z}$ and $m < 0 \Rightarrow -2m-1 \in \mathbb{N}$ and $h(-2m-1) = m$.

* Here we used the (convenient) convention that $\mathbb{N} = \{0, 1, 2, \ldots\}$.

8.a. The range of f is $\mathbb{Z}$.

8.b. The range of g is $[2, \infty)$ as seen by its graph:



8.c. The range of h can be computed by:

1. Knowing that range$(e^x) = [0, e]$ on $(-\infty, 1]$
2. Knowing that range$(\sin^2(x)/2) = [0, \frac{1}{2}]$ on $(-\infty, 1]$
3. Computing the derivative of h.

Points 1,2 tell us that the smallest value of h will be 0.

Point 3 says $h'(x) = e^x \sin^2(x)/2 + 2e^x \sin(x)\cos(x)/2$

$h'(x) = 0 \Rightarrow 0 = e^x \sin(x)(1 + 2\cos(x))$

The derivative is 0 when $\sin(x)=0$: $x = \frac{-\pi}{2}, \frac{-3\pi}{2}, \dots$
and when $1+2\cos(x)=0 \iff \cos(x)=\frac{-1}{2}$: $x = \frac{-2\pi}{3}, \frac{-4\pi}{3}, \dots$

Also knowing that e^x is increasing we make an educated guess that the largest of the x-values that make $h'(x)=0$ will be its max (we also check the critical pts are indeed maxima not minima). And

$$h\left(\frac{-\pi}{2}\right) = e^{-\pi/2}\sin^2(-\pi/2)/2 = e^{-\pi/2}\cdot\frac{1}{2} = \frac{e^{-\pi/2}}{2}$$

8.d. The range of k is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

9. We will prove this by contradiction. Suppose that $0 \neq 1$. Let X be the set of all sets and let $S = \{Y \in X : Y \notin Y\}$. It must be that either $S \in S$ or $S \notin S$. If $S \in S$, then by definition $S \notin S$. If $S \notin S$, then by definition $S \in S$. Since we always get a contradiction, our assumption that $0 \neq 1$ was false. Hence $0 = 1$.