Worksheet 4

2.a.i. $a_0 = 0 \quad a_n = a_{n-1} + 3$ for $n \geq 1$

2.a.ii. $a_0 = 0 \quad a_1 = 3 \quad a_2 = 6 \quad a_n = a_{n-1} + \frac{a_{n-2}}{n-2}$ for $n \geq 3$

2.b. $\sum_{k=1}^{n} (a_k)^2 = \sum_{k=1}^{n} (3k)^2$ Since $a_k = 3k$

$$= 9 \sum_{k=1}^{n} k^2$$

$$= \frac{9n(n+1)(2n+1)}{6}$$ by formula from base.

2.c. Observe that $b_0 = 5 \quad b_1 = 1 \cdot 3 + 2 \cdot 5 + 4$

$$b_2 = 2 \cdot 3 + 2 \cdot (1 \cdot 3 + 2 \cdot 5 + 4) + 4 = 4 \cdot 3 + 4 \cdot 5 + 3 \cdot 4$$

$$b_3 = 3 \cdot 3 + 2 \cdot (4 \cdot 3 + 4 \cdot 5 + 3 \cdot 4) + 4 = 11 \cdot 3 + 8 \cdot 5 + 7 \cdot 4$$

$$b_4 = 4 \cdot 3 + 2(11 \cdot 3 + 8 \cdot 5 + 7 \cdot 4) + 4 = 26 \cdot 3 + 16 \cdot 5 + 15 \cdot 4$$

$$\vdots$$

$$b_n = X_n \cdot 3 + 2^n \cdot 5 + \sum_{i=0}^{n} 2^i \cdot 4$$

Here $X_1 = 1$

$$X_2 = 2 + 2\cdot 1$$

$$X_3 = 3 + 2(2 + 2\cdot 1)$$

$$X_4 = 4 + 2(3 + 2(2 + 2\cdot 1))$$

$$X_5 = 5 + 2(4 + 2(3 + 2(2\cdot 1)))$$

$$= 5 + 2\cdot 4 + 4\cdot 3 + 8\cdot 2$$

$$\vdots$$

$$X_n = n + 2(n-1) + 2^2(n-2) + \cdots + 2^{n-2}\cdot 2$$

$$= \sum_{i=2}^{n} i \cdot 2^{n-i}.$$

3.a.i. $n_1 = \underbrace{26\cdot 4}_{d} + \underbrace{1}_{a} = 105$

3.a.ii. $n_2 = \underbrace{26\cdot 27}_{a} + \underbrace{26\cdot 4}_{d} + \underbrace{1}_{a} = 807$

3.a.iii. $n_3 = \underbrace{26\cdot 27 + 26\cdot 27}_{b} + \underbrace{3}_{c} = 1403$

3.a.iv. $n_4 = \underbrace{2\cdot 27\cdot(26\cdot 27)}_{\text{4th letter } b} + \underbrace{17\cdot(26\cdot 27)}_{\text{3rd letter } r} + \underbrace{2}_{\text{1st letter } b} = 49844$

3.b. The domain of f is the set of letters $\{a, b, \dots, z\}$, and its range is $\{1, \dots, 26\}$.

3.c. We apply the observations from part a. to see that: $k=1$: $n_s = f(s_1)$

$k=2$: $n_s = 26 \cdot f(s_2) + f(s_1)$

$k=3$: $n_s = 26 \cdot 27 \cdot f(s_3) + 26 \cdot (f(s_2)-1) + f(s_1)$

$k=4$: $n_s = 26 \cdot 27^2 \cdot f(s_4)$

$+ 26 \cdot 27 (f(s_3)-1)$

$+ 26 (f(s_2)-1)$

$+ f(s_1)$

Hence probably $n_s = f(s_1) + \sum_{i=2}^{k-1} 26 \cdot 27^{i-2} (f(s_i)-1)$

$+ 26 \cdot 27^{k-2} f(s_k)$.

4.a. sum(map(lambda k: (1-(k+10))* (2(k+10)**2+5)
*(4(k+10)**3 -1),
range(41))))

4.b. L=[(1-k)*(2k**2+5)*(4k**3 -1) for k in
range(10, 51)]

reduce(lambda x,y : x+y , L)

5.a. $$\sum_{i=0}^{7}\sum_{j=0}^{10} ij^2 = \sum_{i=0}^{7} i \sum_{j=0}^{10} j^2$$

$$= \sum_{i=0}^{7} i\left(\frac{10\cdot 11\cdot 21}{6}+1\right)$$ by formula

$$= 386\cdot \sum_{i=0}^{7} i$$

$$= 386\cdot \frac{7\cdot 8}{2}$$ by formula

$$= 10808$$

5.b. $\prod_{i=0}^{7} \sum_{j=0}^{10} ij^2 = \prod_{i=0}^{7} i \sum_{j=0}^{10} j^2 = 0 \cdot \left(\prod_{i=1}^{7} i \sum_{j=0}^{10} j^2 \right) = 0$

5.c. $\sum_{i=0}^{5} \prod_{j=0}^{6} \sum_{k=0}^{7} (i+j+k) = \sum_{i=0}^{5} \prod_{j=0}^{6} \left(\sum_{k=0}^{7} i + \sum_{k=0}^{7} j + \sum_{k=0}^{7} k \right)$

$$= \sum_{i=0}^{5} \prod_{j=0}^{6} \left(8i + 8j + 28 \right) \qquad \frac{7 \cdot (7+1)}{2} = 28$$

$$= \sum_{i=0}^{5} \left(\prod_{j=0}^{6} 8i + \prod_{j=0}^{6} 8j + \prod_{j=0}^{6} 28 \right)$$

$$= \sum_{i=0}^{5} \left((8i)^7 + 0 + 28^7 \right)$$

$$= \sum_{i=0}^{5} 8^7 \cdot i^7 + \sum_{i=0}^{5} 28^7$$

$$= 8^7 \sum_{i=0}^{5} i^7 + 6 \cdot 28^7 .$$

6.a. $\frac{1}{1000} + \frac{1}{1000^2} + \frac{1}{1000^3} + \cdots = 0.001001001\ldots$

6.b. $0.027 = \frac{27}{1000}$ and

$$0.\overline{027} = \sum_{i=0}^{\infty} \left(\frac{27}{1000}\right)\left(\frac{1}{1000}\right)^i$$

$$= \frac{\frac{27}{1000}}{1 - \frac{1}{1000}}$$ by the formula for a geom. progression.

$$= \frac{27}{1000 - 1}$$

$$= \frac{27}{999}$$

$$= \frac{1}{37}$$

6.c. Note that $\frac{1}{64 \cdot 37} = \frac{1}{64} \cdot \frac{\frac{27}{1000}}{1 - \frac{1}{1000}} = \frac{\frac{27/64}{1000}}{1 - \frac{1}{1000}}$

Also note that $\frac{27}{64} = \left(\frac{3}{4}\right)^3 = (0.75)^3 = \left(\frac{75}{100}\right)^3$

Hence $\frac{1}{64 \cdot 37} = \frac{\frac{75^3}{1000000000}}{1 - \frac{1}{1000}}$, so the period is 3, and there are 6 digits before it becomes periodic.

7.a. A set A is countable if there is an injection $A \to \mathbb{N}$.

7.b. Two sets A, B the same cardinality if $|A| = |B|$.

Equivalently, if: 1 there is a bijection $A \to B$

2 there are injections $A \to B$, $B \to A$

3 there are surjections $A \to B$, $B \to A$

7c. Consider the function $f: (0,1) \to (a,b)$ given by $f(x) = x(b-a) + a$. Observe that:

f is injective: if $f(x) = f(y)$, then

$$x(b-a) + a = y(b-a) + a$$

$$x(b-a) = y(b-a)$$

$$x = y$$

f is surjective: if $c \in (a,b)$, then there exists $\varepsilon > 0$, $\varepsilon < b-a$ with $c = a + \varepsilon$.

So $x(b-a) + a = a + \varepsilon \iff x(b-a) = \varepsilon$

$$\iff x = \frac{\varepsilon}{b-a}$$

Since $\varepsilon > 0$ and $\varepsilon < b-a$, $\frac{\varepsilon}{b-a} \in (0,1)$.
Hence f is bijective, and so $|(0,1)| = |(a,b)|$.

7.d. Consider the function $f: \mathbb{N} \to (a,b)$ given by $f(n) = a + \frac{b-a}{2n}$. Observe that:

<u>$f(\mathbb{N}) \subseteq (a,b)$:</u> Since $b > a$, $\frac{b-a}{2n} > 0$,

so $f(n) > a$ for all n. Since n is in the denominator, the largest value of $f(n)$ is for $n=1$, in which case

$$f(1) = a + \frac{b-a}{2} = \frac{2a+b-a}{2} = \frac{b+a}{2} < \frac{b+b}{2} = b.$$

Hence $f(n) < b$ for all n.

<u>f is injective:</u> If $f(n) = f(m)$, then

$$a + \frac{b-a}{2n} = a + \frac{b-a}{2m} \iff \frac{b-a}{2n} = \frac{b-a}{2m}$$
$$\iff 2n = 2m$$
$$\iff n = m,$$

9.a. Since A is countable, there is an injection $f: A \to \mathbb{N}$. Since B is countable, there is an injection $g: B \to \mathbb{N}$. Consider the function:

$$h: A \cup B \longrightarrow \mathbb{N}$$
$$x \longmapsto \begin{cases} 2f(x) & x \in A \\ 2g(x)-1 & x \in B \setminus A \end{cases}$$

We claim this is an injection. First note that $h(A \cup B) \subseteq \mathbb{N}$, as $2n \in \mathbb{N}$ and $2n-1 \in \mathbb{N}$ whenever $n \in \mathbb{N}$. To see h is injective, note that:

$$2f(x) \text{ is even } \quad \forall x \in A$$
$$2g(x)-1 \text{ is odd } \quad \forall x \in B$$

Hence if $h(x) = h(y)$, then it cannot be that $2f(x) = 2g(y)-1$ or $2f(y) = 2g(x)-1$. And if $2f(x) = 2f(y)$, this contradicts f being injective. Similarly $2g(x)-1 = 2g(y)-1$ contradicts g being injective. Hence $A \cup B$ is countable.

9.b. Since $|C| = |\mathbb{R}|$, there exists a bijection $\varphi: C \to \mathbb{R}$. We also know that $|\mathbb{R}| > |\mathbb{N}|$. Consider the function:

$$\begin{aligned} \psi: A \cup C &\to \mathbb{R} \\ a \in A \setminus C &\mapsto 0 \\ c \in C &\mapsto \varphi(c) \end{aligned}$$

This function is surjective because φ is surjective. Hence $|A \cup C| \geq |\mathbb{R}| > |\mathbb{N}|$ as desired.

9.c. Let $F: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the bijection from question 8. Consider the function:

$$\begin{aligned} G: A \times B &\to \mathbb{N} \\ (a, b) &\mapsto F(f(a), g(a)) \end{aligned}$$

This is an injection because all of f, g, F are injections. Hence $A \times B$ is countable.

9.d. Question 9.c. showed that $|A \times B| \leq |\mathbb{N}|$, so we only have to show $|A \times B| \geq |\mathbb{N}|$. We need to assume that A and B are infinitely countable, that is, $|A| = |\mathbb{N}|$ and $|B| = |\mathbb{N}|$. Without loss of generality, assume that $f: A \to \mathbb{N}$ and $g: B \to \mathbb{N}$ are bijections. Then the function G from part 9.c. is a surjection, giving that $|A \times B| \geq |\mathbb{N}|$.

9.e. Consider the function:

$$H : \mathbb{N} \times \cdots \times \mathbb{N} \longrightarrow \mathbb{N}$$

$$(m_1, \ldots, m_n) \longmapsto F(F(\cdots F(m_1, m_2), m_3), \ldots m_n)$$

Since F is a bijection, H is a bijection.