# Worksheet 6

2.a. Suppose that $a|b$ and $c|d$. That means that:

$\exists\, q_1 \in \mathbb{Z}$ s.t. $b = aq_1$

$\exists\, q_2 \in \mathbb{Z}$ s.t. $d = cq_2$

Multiplying these two equations together we get $bd = ac(q_1 \cdot q_2)$. Since $q_1, q_2 \in \mathbb{Z}$, their product is also an integer. This means that $ac|bd$.

2.b. Suppose that $ac|bc$. That means that $\exists\, q \in \mathbb{Z}$ with $bc = qac$. Divide by $c$ to get $b = qa$. This means that $a|b$.

Suppose that $a|b$. That means that $\exists\, p \in \mathbb{Z}$ with $b = pa$. Multiply by $c$ to get $bc = pac$. This means that $ac|bc$.

3.a. $\left\lceil \frac{n}{k} \right\rceil = a : \exists\, r \in \mathbb{Z}\left(\left(0 \leq r < k\right) \wedge \left(n = ak - r\right)\right)$

$\left\lfloor \frac{n}{k} \right\rfloor = b : \exists\, r \in \mathbb{Z}\left(\left(0 \leq r < k\right) \wedge \left(n = bk + r\right)\right)$

3.b. $\left\lceil \frac{n}{k} \right\rceil = a \iff n = ak - r_1$ for some $r_1 \in \{0, \ldots, k\}$

$\left\lfloor \frac{n-1}{k} \right\rfloor + 1 = b \iff n - 1 = (b-1)k + r_2$ for some $r_2 \in \{0 \ldots k\}$

$a = \frac{n + r_1}{k}$

$0 \leq r_1 < k$

$b = \frac{(n-1) - r_2}{k} + 1$ $\quad 0 \leq r_2 < k$

$= \frac{n - 1 - r_2 + k}{k}$

$= \frac{n + (k - r_2 - 1)}{k}.$

Note that $0 \leq r_2 < k$

$0 \geq -r_2 > -k$ multiply by $-1$

$k \geq k - r_2 > 0$ add $k$

$k - 1 \geq k - r_2 > -1$ add $-1$

$k - 1 > k - r_2 \geq 0$ since all are integers

Hence $a = b$.

4.a. If $n$ is even, $n = 2k$ for some $k \in \mathbb{N}$.
Then $n^2 = (2k)^2 = 4k^2 \equiv 0 \cdot k^2 \pmod 4 \equiv 0 \pmod 4$

If $n$ is odd, $n = 2k-1$ for some $k \in \mathbb{N}$.
Then $n^2 = (2k-1)^2 = 4k^2 - 4k + 1$

$$\equiv 0 \cdot k^2 - 0 \cdot k + 1 \pmod 4$$
$$\equiv 1 \pmod 4$$

4.b. Note that $n^4 + n^2 + 1 = n^4 + 2n^2 + 1 - n^2$
$$= (n^2+1)^2 - n^2$$
$$= (n^2 + 1 + n)(n^2 + 1 - n)$$

If $n \geq 2$, then both factors $\geq 2$, so the number is composite.

5.a. $\mathrm{lcm}(a,b) = \dfrac{a \cdot b}{\gcd(a,b)}$

5.b. Let $d = \gcd(a,b)$, so there exist $x, y \in \mathbb{Z}$ with
$d = ax + by$, or $dc = acx + bcy$. $\longrightarrow$

Since $d|a$ and $d|b$, also $dc\,|\,ac$ and $dc|bc$. By question 1 on the homework, it follows that $\gcd(ac, bc) = dc = c \cdot \gcd(a,b)$.

6. Let $d = \gcd(m,n)$, so $\exists\, x, y \in \mathbb{Z}$ with $d = mx + ny$.

Then $\frac{\gcd(m,n)}{n} \binom{n}{m} = \frac{mx+ny}{n} \cdot \binom{n}{m} = \underbrace{\frac{mx}{n}\binom{n}{m}}_{(*)} + \underbrace{y \cdot \binom{n}{m}}_{(**)}$

Note that $(*) = x \cdot \frac{m}{n} \cdot \frac{n!}{m!\,(m-n)!}$

$= x \cdot \frac{(n-1)!}{(m-1)!\,(m-n)!}$

$= x \cdot \frac{(n-1)!}{(m-1)!\,((m-1)-(n-1))!}$

$= x \binom{n-1}{m-1}$

This is an integer, since $\binom{a}{b} \in \mathbb{Z}$ always.

Since $(**)$ is an integer, their sum is too.

7.

| base 2 | base 8 | base 10 | base 16 |
|---|---|---|---|
| 1010101 | 125 | 85 | 55 |
| 11111011111 → →0111110 | 767676 | 257982 | 3EFBE |
| 101100011 → → 00011101 | 261435 | 90909 | 1631D |
| … (long) | 25731043372 | 2942584570 | AF6446FA |

8. $n = 77 = (1001101)_2$.

| x | 1 | 1 | 1 | 7 | 7 | 7 | 7 | 7 | 11 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| power | - | 7 | 7 | 7 | 5 | 5 | 3 | 3 | 3 | 9 | 9 |
| i | - | - | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 2 | 3 |

… and so on.

9.

| x | 31463 | 31463 | 31463 | 9782 | 9782 | 9782 | 2117 |
|---|---|---|---|---|---|---|---|
| y | - | 9782 | 9782 | 9782 | 2117 | 2117 | 2117 |
| r | - | - | 2117 | 2117 | 2117 | 1314 | 1314 |

… and so on.

13. Let $n \in \mathbb{N}$. Then $a^{2n+1} + b^{2n+1}$ always has $a+b$ as a factor. That is,

$$a^{2n+1} + b^{2n+1} = (a+b)\left(a^{2n} + a^{2n-1}b + \cdots\right)$$

Hence for $a = 2$, $b = 1$, $2^{99} + 1 = 2^{99} + 1^{99}$ is divisible by $2 + 1 = 3$, so it is not composite.