Worksheet 7

2.a. Let $P(n)$ be the statement "$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$".

Since $\sum_{k=1}^{1} k = 1 = \frac{1\cdot(1+1)}{2} = \frac{1\cdot 2}{2} = 2$, $P(1)$ is true.

Assume $P(n)$ is true for some $n \geq 1$.

Then $\sum_{k=1}^{n+1} k = n+1 + \sum_{k=1}^{n} = n+1 + \frac{n(n+1)}{2}$ by inductive hypothesis.

$$= \frac{2n+2+n^2+n}{2}$$

$$= \frac{n^2+3n+2}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

Hence $P(n+1)$ is true.

By induction, $P(n)$ is true for all $n \in \mathbb{N}$.

2.b. Let $P(n)$ be the statement "$\sum_{k=1}^{n} 6k^2 = 2n^3 + 3n^2 + n$".

Since $\sum_{k=1}^{1} 6k^2 = 6 = 2 \cdot 1^3 + 3 \cdot 1^3 + 1$, $P(1)$ is true.

Assume $P(n)$ is true for some $n \geq 1$.

Then $\sum_{k=1}^{n+1} 6k^2 = 6(n+1)^2 + \sum_{k=1}^{n} 6k^2$

$= 6n^2 + 12n + 6 + 2n^3 + 3n^2 + n$

$= 2n^3 + 9n^2 + 13n + 6$. $(*)$

Also note that $2(n+1)^3 + 3(n+1)^2 + (n+1)$

$= 2n^3 + 6n^2 + 6n + 2 + 3n^2 + 6n + 3 + n + 1$

$= 2n^3 + 9n^2 + 13n + 6$. $(**)$

Since $(*) = (**)$, $P(n+1)$ is true.

By induction, $P(n)$ is true for all $n \in \mathbb{N}$.

2.c. Let $P(n)$ be the statement "$\sum_{k=1}^{n}(3k-1)(3k+2)=3n^3+6n^2+n$".

Since $\sum_{k=1}^{1}(3k-1)(3k+2)=(3\cdot 1-1)\cdot(3\cdot 1+2)=2\cdot 5=10$, and

$3(1)^3+6(1)^2+1=10$, $P(1)$ is true.

Assume $P(n)$ is true for some $n\geq 1$.

Then $\sum_{k=1}^{n+1}(3k-1)(3k+2)=(3(n+1)-1)(3(n+1)+2)+\sum_{k=1}^{n}(3k-1)(3k+2)$

$$=(3n+2)(3n+5)+3n^3+6n^2+n$$

$$=9n^2+15n+6n+10+3n^3+6n^2+n$$

$$=3n^3+15n^2+22n+10 \quad (*)$$

And also $3(n+1)^3+6(n+1)^2+(n+1)$

$$=3n^3+9n^2+9n+3+6n^2+12n+6+n+1$$

$$=3n^3+15n^2+22n+10 \quad (**)$$

Since $(*)=(**)$, $P(n+1)$ is true.

By induction, $P(n)$ is true for all $n\in\mathbb{N}$.

3.a. Let $P(n)$ be the statement "$2n+1 < 2^n$".

Since $2\cdot 3+1 = 7 < 8 = 2^3$, $P(3)$ is true.

Assume $P(n)$ is true for some $n \geq 3$.

Then $2(n+1)+1 = 2n+3 = 2n+1+2$

$$< 2^n + 2 \quad \text{by inductive hyp.}$$

$$< 2^n + 2^n \quad \text{since } n \geq 3$$

$$= 2^{n+1}$$

Hence $P(n+1)$ is true.

By induction, $P(n)$ is true for all $n \in \mathbb{N}$, $n \geq 3$.

3.6. Let $P(n)$ be the statement "$n! > 2^n$".

Since $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 > 16 = 2^4$, $P(4)$ is true.

Assume $P(n)$ is true for some $n \geq 4$.

Then
$$
\begin{aligned}
(n+1)! &= (n+1) \cdot n! \\
&> (n+1) \cdot 2^n && \text{by inductive hypothesis} \\
&> 2 \cdot 2^n && \text{since } n \geq 4 \\
&= 2^{n+1}
\end{aligned}
$$

Hence $P(n+1)$ is true.

By induction, $P(n)$ is true for all $n \in \mathbb{N}$, $n \geq 4$.

3.c. Let $P(n)$ be the statement "$n^3 \leq 3^n$".

Since $1^3 = 1 \leq 3^1$, $P(1)$ is true.

Assume $P(n)$ is true for some $n \in \mathbb{N}$.

The statement $P(n+1)$ is $(n+1)^3 \leq 3^{n+1}$

$$\Leftrightarrow \quad 1 \leq \frac{1}{(n+1)^3} \cdot 3^{n+1}$$

$$\Leftrightarrow \quad n^3 \leq \left(\frac{n}{n+1}\right)^3 \cdot 3 \cdot 3^n.$$

* If we show $1 \leq \left(\frac{n}{n+1}\right)^3 \cdot 3$, we are done.
But this is false for $n = 1$, $n = 2$.

If $n \geq 3 \Leftrightarrow \frac{1}{n} \leq \frac{1}{3} \Leftrightarrow 1 + \frac{1}{n} \leq \frac{4}{3} \Leftrightarrow \frac{1}{1+\frac{1}{n}} \geq \frac{3}{4}$

Equivalently, $\frac{n}{n+1} \geq \frac{3}{4}$, or $\left(\frac{n}{n+1}\right)^3 \geq \frac{3^3}{4^3}$.
Since $3 \cdot \frac{3^3}{4^3} = \frac{81}{64} > 1$, $P(n+1)$ is true for $n \geq 3$.
$P(2)$ is true: $2^2 = 4 \leq 3^2 = 9$. $P(3)$ is true: $2^3 = 8 \leq 3^3 = 9$.
Hence $P(n)$ is true for all $n \in \mathbb{N}$.