Discrete Structures

Worksheet 9

Spring 2021
March 3/5

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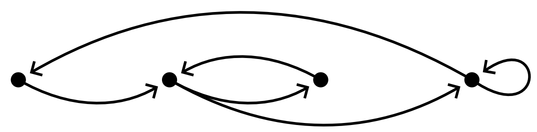
1. **Warm up:** Answer the following questions.

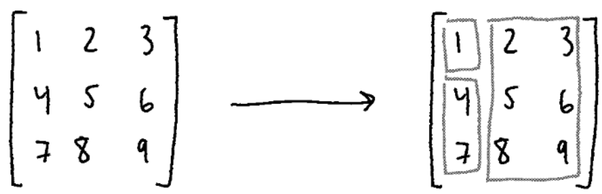
   (a) How many cycles does the following directed graph have?



   (b) Let $\preccurlyeq$ on $\mathbf{N}$ be defined by $a \preccurlyeq b$ iff $a = b - 1$. What is the transitive closure of $\preccurlyeq$?

   (c) True or False: A partition of a set of size n can have at most n classes.

2. Consider the matrix below and the given partition into three rectangles.



   (a) How many different partitions of the same matrix into three rectangles do there exist? Two partitions are *different* if not all of the recatangles have the same size.

   (b) For what natural numbers n do there exist partitions of the matrix into n rectangles? Construct an example for each n.

3. Let $\preccurlyeq$ be a relation on $\mathbf{N}$, defined by $a \preccurlyeq b$ iff $a' = b' - 1$, where $\leqslant$ is the usual order relation on $\mathbf{N}$ and $a \equiv a' \pmod 5$ and $b \equiv b' \pmod 5$.

   (a) Draw arrows on the graph below, with an arrow from i to j if $i \preccurlyeq j$.

   (b) What is the transitive closure of this relation?

   (c) How many equivalence classes does this relation split $\mathbf{N}$ into?

4. Let X be the set of all directed graphs on three vertices a, b, c. In this question, the *size* of a graph will be the number of edges it has.

   (a) In terms of size, how many edges does the smallest such graph have? The largest such graph?

   (b) Let $\sim$ be the relation on X where $G \sim H$ iff G and H have the same size.

      i. How many sets does $\sim$ partition X into?

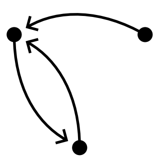
      ii. What are the sizes of these partitions?

   (c) Give an expression for the total number of elements in X (you do not need to evaluate this expression).

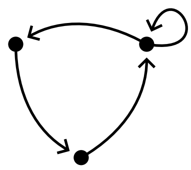
5. Let X be as in Question 4, and let $\leqslant$ be the relation on X where $G \leqslant H$ iff $G = H$ or G is smaller in size than H.

   (a) Show that $\leqslant$ is a partial order on X.

   (b) In this partial order, find a *meet*, or a *lower bound*, and a *join*, or *upper bound*, of the elements



and



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   (c) Show by construction that there is not one unique join nor one unique meet for these two elements.

   (d) Do there exist elements in X for which there is one unique join? For which there is one unique meet?

6. Consider the following sets:

   - $\mathbf{Q}[x]_2 = \{ax^2 + bx + c \ : \ a, b, c \in \mathbf{Q}, a \neq 0\}$ is the set of all quadratic polynomials with rational coefficients
   - W is the set of all real numbers r that are roots for some element of $\mathbf{Q}[x]_2$
   - R is a binary relation on W, where $(r_1, r_2) \in R$ iff $r_1 = r_2$ or r_1 and r_2 are two different roots of the same quadratic equation with rational coefficients

   With these structures, answer the following questions.

   (a) Describe W using set-builder notation.

   (b) Describe R using set-builder notation.

   (c) Show that the relation R is an equivalence relation.

   (d) Find the equivalence classes of R in W and their sizes.

   (e) Describe $W^* \subseteq W$ using set-builder notation so that W^* contains exactly one representative of each equivalence class.

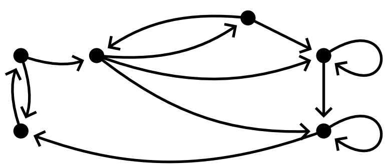
7. **Part 2. Warm up:**

   - Define what is a poset: When is a set A with relation R partially ordered?
   - Apply this definition to the relational digraphs (see the figure for the next problem).
   - Consider two propositions:
     A: "Relation R in set A is a partial order."
     B: "Relation R in set A has Hasse diagram."
   - Will all the relational graphs in the next problem have Hasse diagrams?

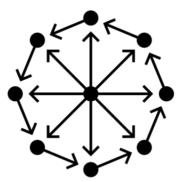
8. This question is about *Hasse diagrams.*

   (a) For each of the following digraphs, find its Hasse diagram.

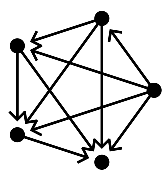
G :



H :



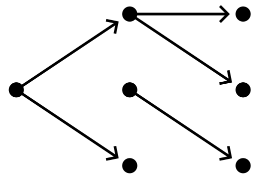
K :



(b) For each of the following Hasse diagrams, add as many edges as you can so that the transitive closure of the diagram remains unchanged.

G :

H :



(c) What is the largest number of edges to add to 9 vertices to get a Hasse diagram?

9. Consider the set $S = \{(a, b) \ : \ a, b \in \mathbf{N}, a + b \leqslant 5\} \subseteq \mathbf{N} \times \mathbf{N}$, and the relation $(a_1, b_1) \leqslant (a_2, b_2)$ whenever $a_1 \leqslant a_2$ and $b_1 \leqslant b_2$.

   (a) Show that $\leqslant$ is a partial order on $\mathbf{N} \times \mathbf{N}$.

   (b) By finding a counterexample, show that $\leqslant$ is not a total order on $\mathbf{N} \times \mathbf{N}$.

   (c) Draw the Hasse diagram of $(S, \leqslant)$.

10. Figure 1 shows some preconditions, how somebody expects to put on various kinds of sportswear.

   (a) Is the diagram a Hasse diagram? If not, what edges can be removed (computed from the others using transitivity)?

   (b) Rearrange all 12 items in such a way that all items have different x coordinates (horizontal offsets); and all the dependency arrows move from the left to the right. This is named *topological ordering* of this poset (also known as DAG - directed acyclical graph).

   (c) Focus on just those 6 items that are circled in red. In how many ways can they be topologically sorted?

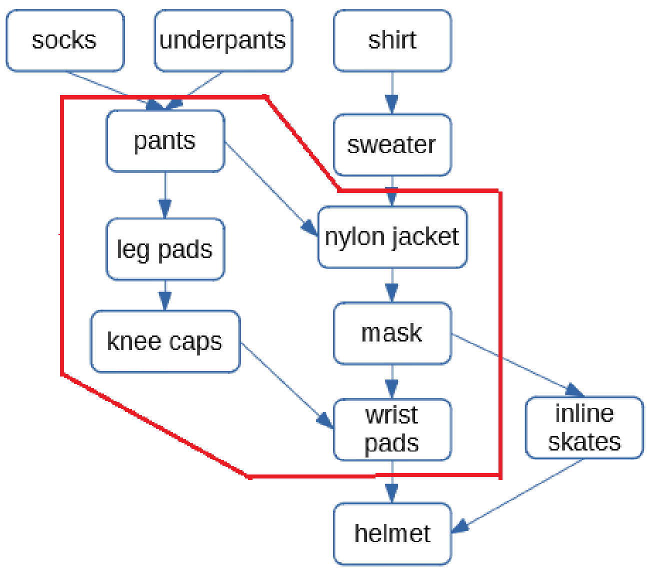


Figure 1: Poset for topological ordering.

11. For statistical processing of data it is common to apply the *melt* operation on a relational table. For example, consider data with grades (Homework, Midterm, Final) for 2 students Jim Ford and Ann Smith:

$R =$

Name	**HW**	**ME**	**FE**
Ann Smith	80	90	95
Jim Ford	95	60	75

.

$S =$

Category
HW
ME
FE

.

After melting this table with respect to the **Name** column, it has just three columns: **Name**, **Category** and **Value**:

$T =$

Name	**Category**	**Value**
Ann Smith	HW	80
Ann Smith	ME	90
Ann Smith	FE	95
Jim Ford	HW	95
Jim Ford	ME	60
Jim Ford	FE	75

(a) Express the melt transformation for this relation using Python Pandas library function `pd.melt`; see `https://bit.ly/3egAYuv`.

(b) Is it possible to express the melted relation T from the relations R and S using 6 operations of relational algebra? (The 6 operations are: Union, difference, Cartesian product, selection, projection and renaming.)

12. Every customer at a supermarket can purchase any of these three items: a pack of diapers *Huggies*; a scented laundry detergent *Tide*; and/or *Bread* (or some combination of the items). The following facts are known:

- $\text{Support}(\{Huggies\}) = 0.05$.
- $\text{Support}(\{Tide\}) = 0.95$.
- $\text{Support}(\{Huggies, Tide\}) = 1.00$.
- $\text{Confidence}(\{Huggies\} \rightarrow \{Bread\}) = 0.90$.
- $\text{Confidence}(\{Tide\} \rightarrow \{Bread\}) = 0.20$.

Assume that there are altogether 1000 transactions in this supermarket.

(a) Are all combinations of items possible? Are there items that are always/never bought together?

(b) Represent the transactions as dots in Euler diagramm with 3 regions *Huggies*, *Tide* and *Bread*.

(c) How many people out of those who bought *Huggies* also bought *Bread*?

(d) How many people out of those who bought *Tide* also bought *Bread*?

(e) What is $\text{Confidence}(\{Bread\} \rightarrow \{Huggies\})$?

Note. Euler diagramm slightly differs from Venn diagram: Empty regions do not need to be shown; one can draw one set inside another, if they are subsets. Euler diagrams can be even more informative, if the areas of regions are roughly proportional to the set sizes.