Worksheet 10.2.

1.a. Let word w = "country"
We can rearrange its letters into $7! = 5040$ strings.
(All letters are unique, so it is a regular permutation)

1.b. Word w = "brittle"
can be rearranged into $\frac{7!}{2!} = 2520$
different strings. It is a permutation where 2 letters "t" are indistinguishable.

1.c. Word w = "popping"
can be rearranged into $\frac{7!}{3!} = 840$
ways. Now 3 letters "p" are the same.

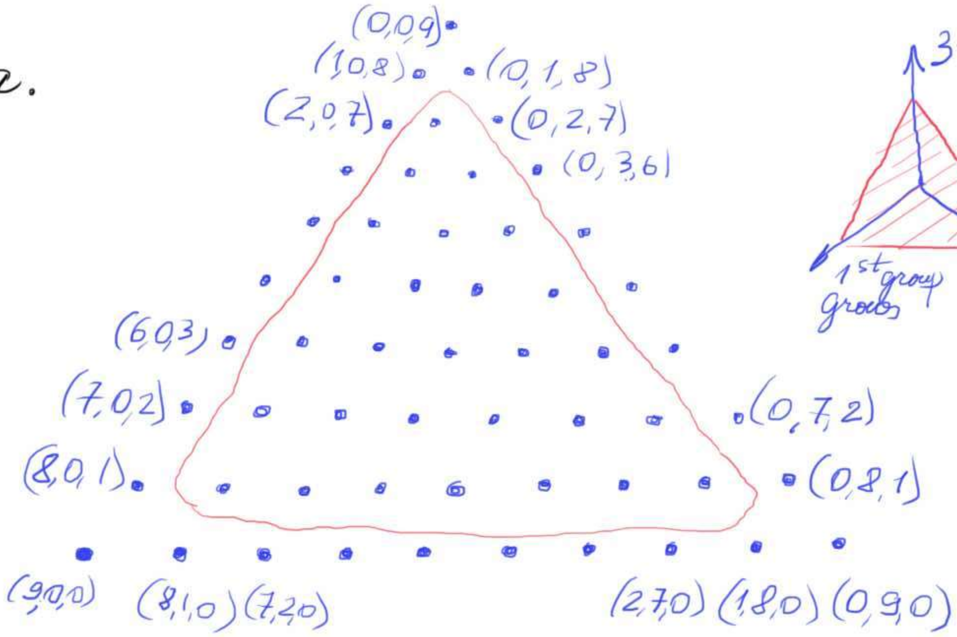
1. d. Word $w =$ "crevice" can be rearranged into $\frac{7!}{2!2!} = 1260$ different ways. It contains two indistinguishable "c"s and two more indistinguishable "e"s.

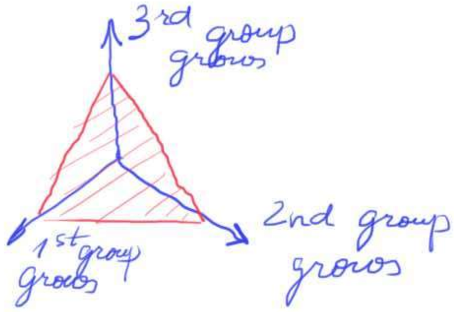
1. e. Word $w =$ "mississippi". The number of unique strings that can be written using all its letters is $\frac{11!}{4!4!2!}$.

$$\begin{cases} 4 \text{ letters "i"} \\ 4 \text{ letters "s"} \\ 2 \text{ letters "p"} \\ 1 \text{ letter "m"} \end{cases}$$

In this case there are multiple groups of indistinguishable letters.

2.a.



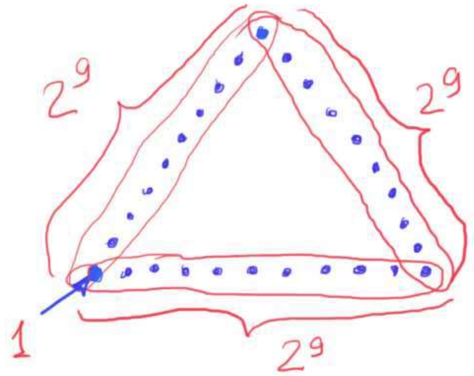


Every point in this diagram represents a way to split 9 people into 3 groups (if we do not care about the individual people assigned to groups). All the points inside the red triangle (i.e NOT on the perimeter) are those where all 3 groups are non-empty.

On the other hand, there are 3^9 ways to assign 9 people to 3 groups. From these

$\binom{9}{0}, \binom{9}{1}, \binom{9}{2}, \dots, \binom{9}{7}, \binom{9}{8}, \binom{9}{9}$ ways

do not use the 3rd group at all.



Their total is 2^9 (bottom side of the triangle).

Other sides are similar.

We get

$$3^9 - 2^9 - 2^9 - 2^9 + 1 + 1 + 1 = 18150$$

2.b. Splitting 9 people into 3 groups of equal size is permutations with repetition. We have 9 "slots" in groups of three and we seat 9 people into these slots:

Group A	Group B	Group C
⊔ ⊔ ⊔	⊔ ⊔ ⊔	⊔ ⊔ ⊔

They can be seated in 9! ways. But the order within the group does not matter, so divide by 3! three times. We get

$$\frac{9!}{3!\,3!\,3!} = 1680$$

Note: If the order of the groups does not matter (groups have <u>no</u> unique names, just the people who end up in the same group) then divide by 3! one more time. $\frac{1680}{3!} = 280$

2.c. In order to expand the expression $(x+y+z)^9$ we can apply the Multinomial theorem:

$$(x+y+z)^9 = \sum_{a+b+c=9} \frac{9!}{a!\,b!\,c!} x^a y^b z^c,$$

where the summation is over all non-negative integers a, b, c such that $a+b+c=9$.

In our case $a=b=c=3$.

So, the coefficient is $\frac{9!}{3!\,3!\,3!} = 1680$.

This is closely related to splitting 9 people into 3 groups in 2.b.

$$\overbrace{(x+y+z)(x+y+z)\cdots(x+y+z)(x+y+z)}^{9 \text{ parentheses}}$$

There are 3^9 ways to pick x's, y's, z's.
Just 1680 of them pick equal # of x, y, z

3.a. There are altogether 365^2 ways to assign birthdays to two people. Out of these there are 365 ways to pick identical birthdays for $n=2$ people. We get the probability $p = \frac{365}{365^2} = \boxed{\frac{1}{365}}$

3.b. There are $365 \cdot 364 \cdot 363$ ways to assign different birthdays to 3 people. Remaining $365^3 - 365 \cdot 364 \cdot 363$ ways will have some overlaps. The probability

$$p = \boxed{\frac{365^3 - 365 \cdot 364 \cdot 363}{365^3}} \approx \frac{3}{365}$$

3.c. By Pidgeonhole principle $n = \boxed{366}$ people will always have overlapping birthdays.

4.a. Use Binomial Formula:

$$(x+1)^4 = \binom{4}{0} x^4 1^0 + \binom{4}{1} x^3 1^1 + \binom{4}{2} x^2 1^2 + \binom{4}{3} x^1 1^3 + \binom{4}{4} x^0 1^4 =$$

$$= x^4 + 4x^3 + 6x^2 + 4x + 1.$$

4.b. $$(x+y)^{20} = \ldots + \underbrace{\binom{20}{9} x^{11} y^9 + \binom{20}{10} x^{10} y^{10} + \binom{20}{11} x^9 y^{11}} + \ldots$$

Just look at the middle 3 terms (out of all 21 terms).

These can be evaluated manually:

$$\binom{20}{9} = \frac{20!}{11!\,9!} = \frac{\cancel{12}\cdot 13\cdot \cancel{14}\cdot \cancel{15}\cdot \overset{2}{\cancel{16}}\cdot 17\cdot \cancel{18}\cdot 19\cdot 20}{1\cdot \cancel{2}\cdot \cancel{3}\cdot \underset{\cancel{2}}{\cancel{4}}\cdot \cancel{5}\cdot \cancel{6}\cdot \cancel{7}\cdot \cancel{8}\cdot \cancel{9}} =$$

$$= 13\cdot 2\cdot 17\cdot 19\cdot 20 = 442\cdot 380 = 167960.$$

Typically, we evaluate these using a calculator:

$$(x+y)^{20} = \ldots + 167960\,x^{11}y^9 +$$

$$+ 184756\,x^{10}y^{10} + 167960\,x^9 y^{11} + \ldots$$

Anaconda Powershell Prompt (Anaconda3)

```
(base) PS C:\> python
Python 3.8.5 (default, Sep
Type "help", "copyright", "
>>> import math
>>> math.comb(20,9)
167960
>>> math.comb(20,10)
184756
>>> math.comb(20,11)
167960
>>>
```

4.c. $(2x+5y)^{10}=\sum_{k=0}^{10}\binom{10}{k}(2x)^{10-k}(5y)^k$

In the sequence $2^{10-k}\cdot 5^k$ every next member is $\frac{5}{2}$ times bigger than the previous one.

```
>>> import math
>>> list(map(lambda x: math.comb(10,x), range(0,11)))
[1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1]
```

As we can see from the 10^{th} row in the Pascal's triangle $\binom{10}{7}=120$ reduces more than $\frac{5}{2}$ times as it is replaced by $\binom{10}{8}=45$.

So, the biggest coefficient is

$\binom{10}{7}2^3\cdot 5^7=120\cdot 2^3\cdot 5^7=75\,000\,000.$

4.d. $(x+y^2)^{12} = \dots + \binom{12}{8} x^8 (y^2)^4 + \dots =$

$= \dots + \boxed{495}\, x^8 y^8 + \dots$.

(Other terms don't have $x^8 y^8$).

4.e. $\left(x+\frac{1}{x}\right)^8 = \sum_{k=0}^{8} \binom{8}{k} x^{8-k} \left(\frac{1}{x}\right)^k$.

The expansion contains 9 terms with these powers: $\underbrace{x^8, x^6, x^4, x^2, 1, x^{-2}, x^{-4}, x^{-6}, x^{-8}}$.

The coefficient for the free term 1 (or $x^4 \left(\frac{1}{x}\right)^4$) is $\binom{8}{4} = \boxed{70}$

4.f. Substitute the 6^{th} row in Pascal's triangle

$\binom{6}{0}=1$, $\binom{6}{1}=6$, $\binom{6}{2}=15$, $\binom{6}{3}=20$, $\binom{6}{4}=15$, $\binom{6}{5}=6$, $\binom{6}{6}=1$

$$(x^6+6x^5+15x^4+20x^3+15x^2+6x+1)^2 =$$

$$= \ldots + x^6\cdot 1^6 + (6x^5)(6x) + (15x^4)(15x^2) + (20x^3)(20x^3) +$$

$$+ (15x^2)(15x^4) + (6x)(6x^5) + 1\cdot x^6 =$$

$$= \boxed{(1^2+6^2+15^2+20^2+15^2+6^2+1^2)}x^6 =$$

$$= \boxed{924}x^6$$

From another side, this equals

$$((x+1)^6)^2 = (x+1)^{12} = \ldots + \boxed{\binom{12}{6}}x^6 + \ldots$$

This latter coefficient also evaluates to the same $\binom{12}{6} = \frac{12!}{6!\,6!} = \boxed{924}$

4.g. given $f(x) = (x+1)^5$, find $f'(1)$.

We have $f'(x) = 5(x+1)^4$.

Substitute $x=1$, get $\boxed{f'(1) = 5 \cdot 2^4 = 80}$.

On another hand, transform:

$$f(x) = \binom{5}{0}x^5 + \binom{5}{1}x^4 + \binom{5}{2}x^3 + \binom{5}{3}x^2 + \binom{5}{4}x + \binom{5}{5}.$$

$$f'(x) = 5\binom{5}{0}x^4 + 4\binom{5}{1}x^3 + 3\binom{5}{2}x^2 + 2\binom{5}{3}x + \binom{5}{4}.$$

$$f'(1) = 5\underbrace{\binom{5}{0}}_{=1} + 4\underbrace{\binom{5}{1}}_{=5} + 3\underbrace{\binom{5}{2}}_{=10} + 2\underbrace{\binom{5}{3}}_{=10} + \underbrace{\binom{5}{4}}_{=5}$$

We get an identity

$$\boxed{5 \cdot 2^4 = 5\binom{5}{0} + 4\binom{5}{1} + 3\binom{5}{2} + 2\binom{5}{3} + \binom{5}{4}}$$

or numerically:

$$80 = 5 \cdot 1 + 4 \cdot 5 + 3 \cdot 10 + 2 \cdot 10 + 5$$

5.a. Expand $(1-1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k}(-1)^k$.

$$0 = \binom{2n}{2n} - \binom{2n}{2n-1} + \binom{2n}{2n-2} - \binom{2n}{2n-3} + \ldots - \binom{2n}{1} + \binom{2n}{0}$$

Rewrite the negative terms on the opposite side:

$$\boxed{\binom{2n}{n} + \binom{2n}{2n-2} + \ldots + \binom{2n}{2} + \binom{2n}{0} = \binom{2n}{2n-1} + \binom{2n}{2n-3} + \ldots + \binom{2n}{1}}$$

5.b. $\sum_{i=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$. (Should prove this)

Evaluate the derivative $((x+1)^n)'$ at $x=1$.

In 4.g. we did this for $n=5$.

We got this identity:

$$5 \cdot 2^4 = 5\binom{5}{0} + 4\binom{5}{1} + 3\binom{5}{2} + 2\binom{5}{3} + \binom{5}{4}.$$

Since $\binom{5}{k} = \binom{5}{5-k}$, we can rewrite

$$5 \cdot 2^4 = 5 \cdot \binom{5}{5} + 4 \cdot \binom{5}{4} + 3 \cdot \binom{5}{3} + 2 \cdot \binom{5}{2} + \binom{5}{1}$$

or $\boxed{5 \cdot 2^{5-1} = \sum_{k=1}^{5} k \cdot \binom{5}{k}}$

For other values n the proof is identical.

5.c. Expand $((x+1)^n)^2 = (x+1)^{2n}$ in two different ways and find the coefficient for x^n.

Expanding $(x+1)^{2n}$ we get

$$\sum_{k=0}^{2n} \binom{2n}{k} x^k = \ldots + \binom{2n}{n} x^n + \ldots$$

On the other hand,

$$\left(\binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \ldots + \binom{n}{n-1}x + \binom{n}{n}\right)^2.$$

~~If~~ we square this polynomial, we get various terms with power x^n:

$$\binom{n}{0}\cdot\binom{n}{n} + \binom{n}{1}\cdot\binom{n}{n-1} + \ldots + \binom{n}{n-1}\binom{n}{1} + \binom{n}{n}\binom{n}{0}.$$

Since $\binom{n}{0} = \binom{n}{n}$; $\binom{n}{1} = \binom{n}{n-1}$, we get

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n-1}^2 + \binom{n}{n}^2.$$

(See also 4.f where $n=6$).

6.a. $H = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$; $V = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$.

We multiply these permutations

$$V \circ H = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}}$$

Note that in $V \circ H$ we apply H before V

$$H \circ V = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}}$$

We see that $V \circ H = H \circ V$.

6.b. $M_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$, $M_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$

$$M_2 \circ M_3 = M_3 \circ M_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

6.c. $V \circ M_2 = \boxed{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}}$, but

$M_2 \circ V = \boxed{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}}$. So $V \circ M_2 \neq M_2 \circ V$.

7. a. Represent every piece of candy by an asterisk (*) and insert vertical bars whenever we switch from Candy Type A to Type B or Type C

$$\underbrace{* * * *}_{\text{Type A}} \Big| \underbrace{* * * * * * *}_{\text{Type B}} \Big| \underbrace{* * * * * * * * *}_{\text{Type C}}$$

The diagram represents 4 candies A, 7 candies B and 9 candies C.

There are $\boxed{\binom{22}{2} = 231}$ ways to insert 2 bars (and 20 asterisks) in 22 slots. This is the known formula of combinations with repetition: $\binom{n+r-1}{n-1}$ where we pick $r=20$ elements out of $n=3$ kinds (with repetition)

7.b. $x_1 + x_2 + x_3 + x_4 = 17$
can be solved using same asterisk-bar diagrams as 7.a.
(17 asterisks *, 4-1=3 dividing bars |)
It is also combination with repetitions: Pick $r=17$ items out of $n=4$ different kinds (the order of items does not matter, only the four counts do.
We get $\boxed{\binom{n+r-1}{n-1} = \binom{20}{3} = 1140}$.

7.c. Filling 3 chairs out of 20 (where the filling order does not matter) is regular combinations:
$\binom{20}{3} = \frac{20!}{17!\,3!} = \boxed{1140}$

7.d. Let us solve a different problem first: We have 10 empty chairs  and 5 inseparable pairs  of two chairs (the right one is occupied by a person, but to the left there is an empty chair that serves as a padding. There are $\binom{15}{5} = 3003$ ways to mix 10 empty chairs and 5 pairs.

But in reality someone can sit on the leftmost chair as well. Then we have 4 pairs and 11 empty chairs remaining: $\binom{16}{4} = 1820$.

Altogether $\binom{15}{5} + \binom{16}{4} = \boxed{4823}$ ways.

8.a. There are $\binom{100}{51} \approx 9{,}9 \cdot 10^{28}$ ways to pick 51 numbers; no chance to analyze each variant separately.

```
>>> import math
>>> math.comb(100,51)
98913082887808032681188722800
```

8.b. An easy way to ensure that two numbers are mutually prime: Pick them next to each other

$\underbrace{\boxed{n \quad n+1}}$ We can separate all numbers $1, 2, 3, \ldots, 99, 100$ into such buckets:

1 bucket

$$\underbrace{(1;2), (3;4), (5;6), \ldots, (97;98), (99;100)}_{50 \text{ buckets}}$$

8.c. Whenever one picks 51 numbers, at least 2 will be in the same bucket (Pidgeonhole principle) so they are mutually prime

BTW, one can pick 50 numbers (all even ones) so that none are relative primes.