Discrete Structures

Worksheet 11.2

Spring 2021
March 19

---

1. **Warm up:** There are 6 doors in a Monty Hall game. A prize (worth 1 EUR) is behind one of the doors (the remaining 5 doors do not have any prize, so opening them gives 0 EUR). A participant randomly chooses some 3 doors: d_1, d_2, d_3. Let X_1, X_2, X_3 be random variables describing the prize in EUR behind the doors d_1, d_2, d_3 respectively.

   (a) What is the probability to win 1 EUR? Namely, find $p(X_1 + X_2 + X_3 = 1)$.

   (b) Find the conditional probabilities:
   $p(X_2 = 1 \mid X_1 = 0)$ and $p(X_3 = 1 \mid X_1 = 0 \wedge X_2 = 0)$. Describe them informally.

   (c) Find the expected values $E(X_1)$, $E(X_1 + X_2)$ and $E(X_1 + X_2 + X_3)$.

   (d) Find the variances $V(X_1)$, $V(X_2)$, $V(X_3)$.

   (e) Find the variance $V(X_1 + X_2 + X_3)$. Is it three times larger than $V(X_1)$?

   (f) Find the variance $V(X_1 + X_2 + X_3)$, if the prize is measured in eurocents (not euros).

2. Some gambling parlor offers the following game: A visitor chooses any number $n \in \{1, 2, 3, 4, 5, 6\}$. Then he rolls three fair dice.
   If one of the dice rolls equals n, then the visitor wins 1 EUR.
   If two of the dice rolls equal n, then the visitor wins 2 EUR.
   If three of the dice rolls equal n, then the visitor wins 3 EUR.
   If none of the dice rolls equals n, then the visitor loses 1 EUR.
   Let X be the random variable describing the money that the visitor wins during one round of this game (one round = 3 dice rolls).

   (a) Find $p(X = 1)$,
   (b) Find $p(X = 2)$,
   (c) Find $p(X = 3)$,
   (d) Find $p(X = -1)$,
   (e) Find $E(X)$. What is the conclusion about this game: Who wins money (and how fast) as it is played long enough?

3. Urn A has three red and five white balls and Urn B has two red and seven white. You pick an urn at random and draw a red ball from it. What is the probability that it was Urn A?

4. You are looking for biased coins at a mint. You know that 99 out of every 100 coins are perfectly fair (probability to get heads is 50%) and that 1 out of 100 coins lands on heads 60% of the time.

   You flip a coin 50 times and get 33 heads. What are the odds that this coin is biased?

5. In some imagined country there are 114 boys born for every 100 girls. Assume that some couple decides to have children until they have a girl. Denote by X the number of children in this family. It is a random variable that can take values $1, 2, 3, \ldots$ ($X = 1$, if they have a girl as their first child, etc.)

   (a) Find p – the probability that a girl is born.

(b) Is it possible that this couple will have infinitely many children? What is the probability of this event?

(c) Find $E(X)$ – the expected number of children in this family.

(d) Find $V(X)$ – the variance of the number of children in this family.

Note. You may need to look up the formulas of Geometric distributions; see `https://bit.ly/3cRuEHp` and also (Rosen2019, p.510), Chapter 7.4.5 *The Geometric Distribution.*

6. Suppose that the number of aluminum cans recycled in a day at a recycling center is a random variable with an expected value of 50000 and a variance of 10000.
Use Chebyshev's inequality to provide a lower bound on the probability that this center will recycle 40000 to 60000 cans on a certain day.

Note. Chebyshev's inequality estimates the probability that the value of the random variable X will be further than $k\sigma$ away from the expected value $\mu = E(X)$ (where $\sigma^2 = V(X)$ is the variance). See `https://bit.ly/3tETO2u` and also (Rosen2019, p.517) textbook, Chapter 7.4.8.