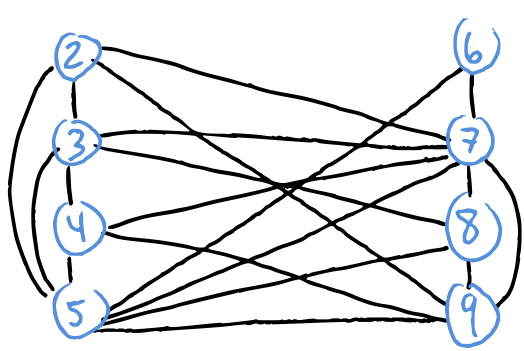
Worksheet 13

2.a. The statement "a and b are coprime" is $\gcd(a,b)=1$. Since $\gcd(a,b)=\gcd(b,a)$, this statement (and so the relation R) is symmetric.

The relation R is not reflexive because $\gcd(2,2)\neq 1$.

The relation R is not transitive because $\gcd(2,3)=1$ and $\gcd(3,4)=1$, but $\gcd(2,4)\neq 1$.

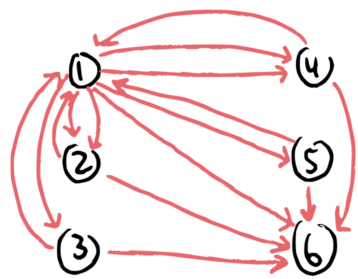
2.b.



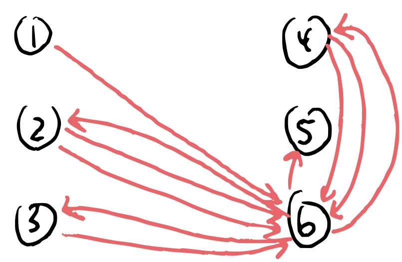
2.c. Use 2, 4, 8 as one partition and 3, 5, 7 as another. Every number in each is coprime to every number in the other.

3.a. $M_G = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad M_H = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

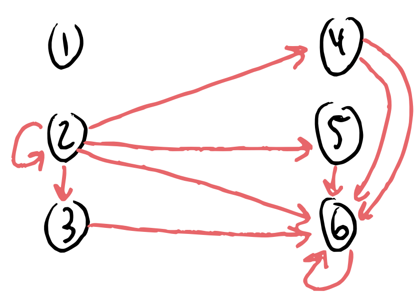
3.b. $M_G \cdot M_H = \begin{bmatrix} 0 & 2 & 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$



$M_H \cdot M_G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$



$M_H^T \cdot M_G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$



4.a. If two graphs are isomorphic, they must have the same number of edges. These graphs have 12 and 10 edges, respectively, so they cannot be isomorphic.

4.b. If two graphs are isomorphic, their vertices must have the same degrees. The graph on the left has vertices with degrees 2, 3 whereas the graph on the right has vertices with degrees 1, 2, 3, 4, so they cannot be isomorphic.

4.c. For $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, let $\varphi: V_1 \to V_2$ be defined as:

$\varphi(a) = a$	$\varphi(f) = b$
$\varphi(b) = i$	$\varphi(g) = j$
$\varphi(c) = e$	$\varphi(h) = f$
$\varphi(d) = h$	$\varphi(i) = g$
$\varphi(e) = d$	$\varphi(j) = c$

We may check that this induces a bijection on edges as well.

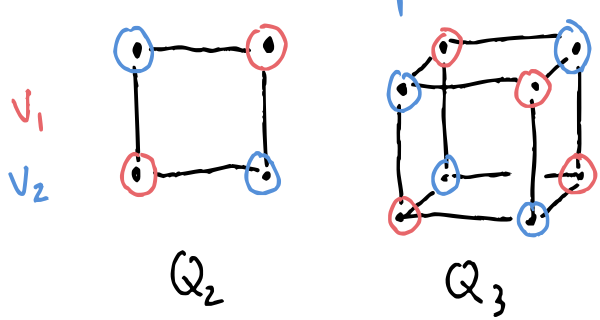
5.a. For $G=(V,E)$, since $G = K_{n,n/2}$, it follows that:

$$|V| = n + \frac{n}{2} = \frac{3n}{2} \qquad |E| = n \cdot \frac{n}{2} = \frac{n^2}{2}$$

The density of G is $\dfrac{2|E|}{|V|(|V|-1)} = \dfrac{2 \cdot \frac{n^2}{2}}{\frac{3n}{2}\left(\frac{3n}{2}-1\right)} = \dfrac{2n^2}{3n(3n-2)}$

$$= \frac{2n}{9n-4}.$$

5.b. Recall that the vertices of Q_n are binary strings of "0" and "1" of length n, and two vertices are connected if the strings differ by a single letter. For $n=2$ and $n=3$ the partitions are clear:



For Q_n the bipartition can be defined inductively.

5.c. If $G=(V,E)$ is bipartite, then $V=V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ and $|V_1|=a$, $|V_2|=b$. That is,

$$|V| = a+b \qquad |E| = ab$$

Hence $|E| = a(|V|-a) = a|V| - a^2$. As a function of a, $|E|$ has a local extremum at $\frac{a}{2}$, as:

$$0 = |E|'(a) = |V| - 2a \implies a = \frac{|V|}{2}$$

This is a local max, so the largest number of edges G can have is when $a = \frac{|V|}{2}$, or $|E| = \frac{|V|}{2} \cdot \frac{|V|}{2} = \frac{|V|^2}{4}$. Hence $|E| \le \frac{|V|^2}{4}$ in general.

6. By the handshaking theorem:

$$2|E| = \sum_{v \in V} \deg(v) = \sum_{v \in V} k = |V| k$$

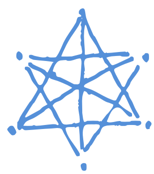
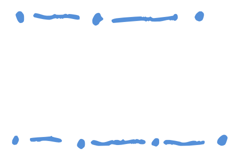
Since k is odd, k does not divide 2. Hence k divides $|E|$.

7a. Call the graphs G, H, K from left to right.

G has 2 automorphisms

H has $5!$ automorphisms

K has 1 automorphism

7.b. $\overline{C_6} =$  $\overline{K_{3,4}} =$ 

$\overline{Q_4}$ is hard to draw. $\overline{K_5} =$