Discrete Structures

Worksheet 13

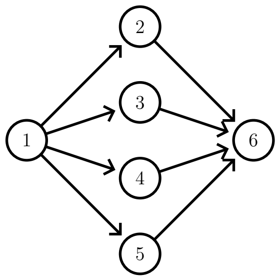
Spring 2021
March 31

1. **Warm up:** Answer the following True / False questions for a graph $G = (V, E)$.
   (a) Every graph is the union of finitely many simple graphs.
   (b) Every edge in a tree is a cut edge.
   (c) If G is undirected, $|V| = 3$, and $|E| = 3$, then G must be connected.
   (d) If G is directed, $|V| = 3$, and $|E| = 3$, then G must be connected.

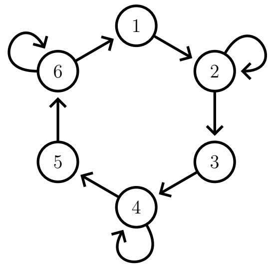
2. Let aRb be relation given by "a and b are coprime" for $a, b \in \mathbf{N}$.
   (a) Show that R is symmetric. Give examples that show R is not reflexive or transitive.
   (b) Construct a graph $G = (V, E)$ where $V = \{2, \ldots, 9\}$ and $\{a, b\} \in E$ whenever aRb.
   (c) Find $K_{3,3}$ as a subgraph of G.

3. Consider the directed graphs G and H:

G =

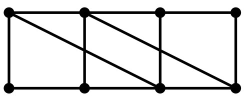


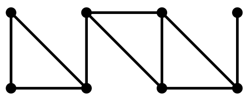
H =



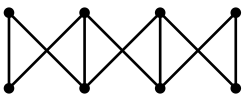
   (a) Construct the adjacency matrices M_G and M_H for the graphs.
   (b) Compute the matrices $M_G \cdot M_H$, $M_H \cdot M_G$ and $M_H^T \cdot M_G$.
   (c) Construct the directed graphs from the matrix products of part (b).

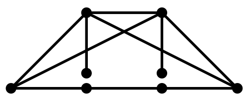
4. This question is about *isomorphisms*.
   (a) Using an argument about edges, explain why the following graphs are not isomoprhic.





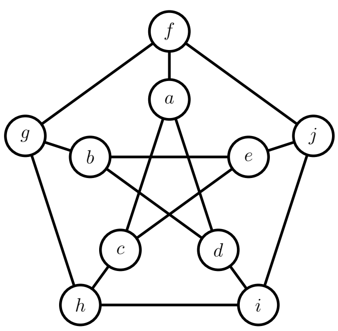
   (b) Using an argument about degrees, explain why the following graphs are not isomoprhic.



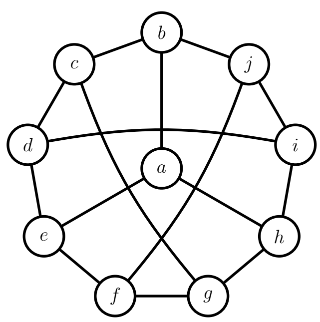


   (c) Describe explicitly an isomorphism between the following two graphs.

G_1 =



G_2 =



5. This question is about *bipartite* graphs. Let $G = (V, E)$ be a bipartite graph.

   (a) Compute the density of G if $G = K_{n,n/2}$ for $n \in \mathbf{N}$ even.

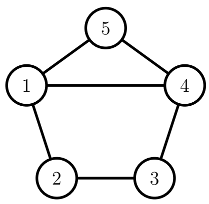
   (b) Show by construction that Q_n is bipartite.

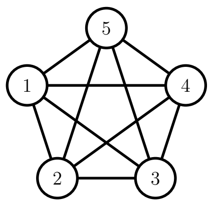
   (c) Prove that $|E| \leqslant \frac{|V|^2}{4}$.

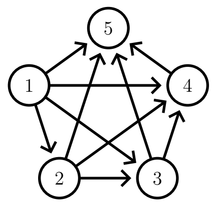
6. This question is about the *handshaking theorem*. Let $G = (V, E)$ be a k-regular graph, where k is an odd number. Prove that the number of edges in G is a multiple of k.

7. This question is about *automorphisms* and the *complement* of a graph $G = (V, E)$. An isomorphism f from G to H is an *automorphism* if $G = H$. The *complement* $\overline{G} = (V, \overline{E})$ has the same vertices and $e \in \overline{E}$ iff $e \notin E$.

   (a) How many automorphisms do each of the following graphs have?







   (b) Compute the complements of the following graphs.

   i. C_6 ii. $K_{3,4}$ iii. Q_4 iv. K_5

   (c) For g undirected, prove that f is an automorphism of G iff f is an automorphism of $\overline{G}$. Is the claim true if G is directed?